Variance Swaps

MS&E 345 Advanced Topics in Financial Engineering

Winter '09

Gerry Tsoukalas [gts@stanford.edu]
Xianyi Zeng [rhodusz@stanford.edu]

Outline of Presentation

Part I: Background

- Introduction to Variance Swaps
- Pricing Intuition
- The Variance Swap Market

Part II: Replication and Pricing

Part III: Variance Swap Strategies

Outline of Presentation

Part I: Background

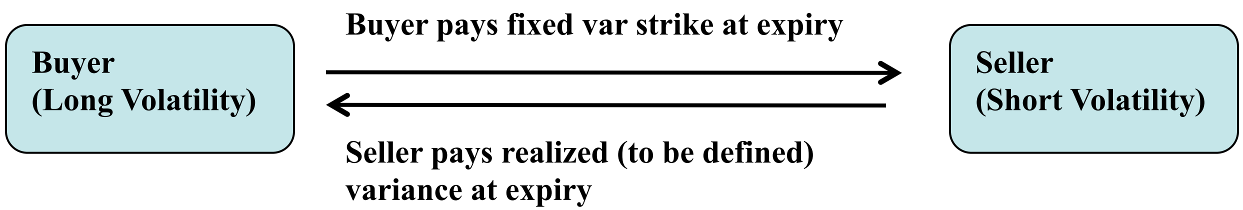
>>> • Introduction to Variance Swaps
• Pricing Intuition
• The Variance Swap Market

Part II: Replication and Pricing

Part III: Variance Swap Strategies

Introduction to Variance Swaps

- The Product
  - Offers direct and “simple” exposure to the volatility of an underlying asset



  - Liquid across major equity indices and some large-cap stocks
  - Used for various purposes:
    - Take a volatility view (Long or Short)
    - Diversify returns
    - Trade forward volatility, correlation, dispersion
  - Replication:
    - Exact replication by an infinite continuous portfolio of vanilla options
    - In practice, hedged with a “small” number of options
    - Pricing reflects volatilities across the entire skew surface
    - In practice, VSwaps trade at a slight premium to ATM implied volatilities

Introduction to Variance Swaps

- The VSwap contract
  - OTC product: Two parties agree to enter into a swap with maturity T
    - The buyer of the swap receives realized variance, σ^2, over the life of the contract at date T
    - The seller of the swap receives a fixed pre-determined strike K^2 at date T. The strike reflects market estimates of future volatility (implied volatility) at time t.

$$payoff_{(long)} = h(T) = N_{Var}(\sigma^2 - K^2) = \frac{N_{Vega}}{2K}(\sigma^2 - K^2)$$

N_{Vega} represents the average profit/loss for a 1% (1 vega) change in volatility

- Measuring realized variance and volatility
  - Issues
    - Frequency of sampling: hourly? Daily? Intraday? Weekly? Time-period?
  - Actual method used: "RMS" (Root-Mean-Squared) = ignore mean
    - Simplifies calculation (a little), error made not too big, mean is typically around zero

$$\sigma^2 = \frac{252}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{S_i}{S_{i-1}}\right)\right]^2$$

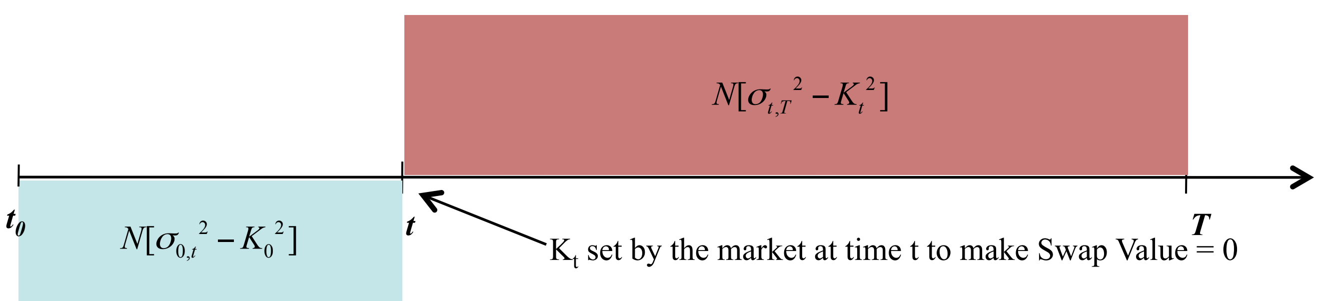
where S_i is the price of the underlying at closing and N is the number of trading days during the length of the contract.

VSwap Mark-to-Market

- Variance is additive, which simplifies the MTM, we need:
  - Realized variance since the start of the swap
  - Implied variance (new Strike) from *t* until expiry *T*
  - Additivity equation:

$$(T-t_0)\,\mathrm{var}_{t_0\to T}(S) = (t-t_0)\,\mathrm{var}_{t_0\to t}(S) + (T-t)\,\mathrm{var}_{t\to T}(S) \Rightarrow \begin{cases} t_0 = 0 \\ \mathrm{var}_{t_0\to T}(S) = \sigma_{0,T}^{2} \\ \mathrm{var}_{t_0\to t}(S) = \sigma_{0,t}^{2} \\ \mathrm{var}_{t\to T}(S) = \sigma_{t,T}^{2} \end{cases} \Rightarrow \begin{cases} \sigma_{0,T}^{2} = \lambda\sigma_{0,t}^{2} + (1-\lambda)\sigma_{t,T}^{2} \\ \text{with } \lambda = \dfrac{t}{T} \end{cases}$$

$$Payoff(0,T) = N[\sigma_{0,T}^{2} - K_0^{2}] \Rightarrow \boxed{PV(t,T) = Ne^{-r(T-t)}\{[\sigma_t^{2} - K_0^{2}]\lambda + [K_t^{2} - K_0^{2}](1-\lambda)\}}$$



Outline of Presentation

Part I: Background

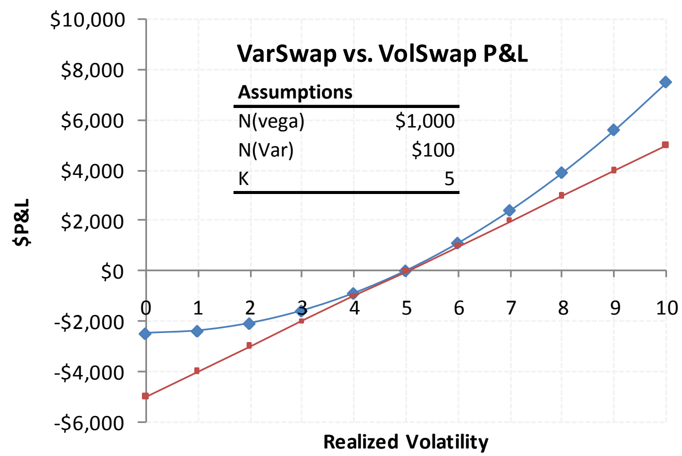
- Introduction to Variance Swaps
- >>> Pricing Intuition
- The Variance Swap Market

Part II: Replication and Pricing

Part III: Variance Swap Strategies

Pricing Intuition

- The strike K, reflects the market's best guess of future volatility but with a premium (actually 2 premiums)
  – Convexity premium:
    - Variance swaps are convex in volatility: The gain from an increase in volatility is greater in absolute terms than the loss from the corresponding decrease
    - To take this into account, traders charge a premium to the ATM implied volatilities



		P&L	
Real. Vol	**Real. Var**	**VarSwap**	**VolSwap**
0	0	-$2,500	-$5,000
1	1	-$2,400	-$4,000
2	4	-$2,100	-$3,000
3	9	-$1,600	-$2,000
4	16	-$900	-$1,000
5	**25**	**$0**	**$0**
6	36	$1,100	$1,000
7	49	$2,400	$2,000
8	64	$3,900	$3,000
9	81	$5,600	$4,000
10	100	$7,500	$5,000

  – Volatility risk premium (replication premium):
    - Theoretical price calculated from prices of replicating options, so the strike K can be thought of a weighted average of vanilla option implied vols. In the presence of skew and skew convexity, avg. vols will usually be above ATM vol, making the VSwap more expensive.
- VSwaps usually trade 1-2 vegas above ATM volatility

Outline of Presentation

Part I: Background

- Introduction to Variance Swaps
- Pricing Intuition
- >>> The Variance Swap Market

Part II: Replication and Pricing

Part III: Variance Swap Strategies

The Variance Swap Market

- First mentioned in the 1990's – took off early 2000's
  - Mostly on index underlyings, EURO STOXX 50, S&P 500
  - Also on large-cap constituents allowing for dispersion trades
- Steady growth over past few years
  - Marketed as an alternative to options without path dependence issues and transaction costs resulting from delta-hedging
- Significant increases in liquidity
  - Variance swaps moved from exotics desks into flow-trading
  - Bid/offer spreads on indices at around 0.4 vegas, 1-2 on single-names
  - Liquid maturities ranging from 3 months to 2 years
  - VIX represents theoretical prices of VSwaps on S&P
  - Around 30% of the vega traded in the market is done so via Vswaps
- Less liquidity in other assets (bonds, fx, commodities)
  - In theory, this should change

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

- Delta Hedging in Black-Scholes
- Replicating a Variance Swap
- The Fair Strike
- Numerical Application

Part III: Variance Swap Strategies

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

>>>
- Delta Hedging in Black-Scholes
- Replicating a Variance Swap
- The Fair Strike
- Numerical Application

Part III: Variance Swap Strategies

Delta Hedging in Black-Scholes

- P/L over small price movement, in terms of gamma

$$[C(S+dS) - C_S(S)(S+dS)] - [C(S) - C_S(S)S]$$

$$= C_S dS + \frac{1}{2} C_{SS} dS^2 - C_S dS + O(dS^3)$$

$$\approx \frac{1}{2} C_{SS} dS^2 = \frac{1}{2} \Gamma dS^2$$

- Dollar gamma
  - Dollar gamma measure P/L in terms of return

$$\$\Gamma = \Gamma S^2 / 100$$

  - P/L in terms of dollar gamma: connection to return

$$P/L = \frac{1}{2} \Gamma dS^2 = \Gamma S^2 \left(\frac{dS}{S} \right)^2 = 50\$\Gamma R^2$$

Delta Hedging in Black-Scholes

- There ain't no such thing as a free lunch
  - At the first glance, delta hedge always give positive P/L
  - We have only considered asset price's movement, omitting the theta

$$\theta = C_t = -\frac{1}{2}\Gamma S^2\sigma^2 \qquad \text{(We assume zero risk free rate)}$$

  - P/L with theta

$$P/L = \frac{1}{2}\Gamma dS^2 + \theta dt = \frac{1}{2}\Gamma S^2[R^2 - \sigma^2 dt] = 50\$\Gamma[R^2 - \sigma^2 dt]$$

- Realised volatility
  - What is return: $1+R=(S+dS)/S=S_{dt}/S_0$
  - What is realised volatility: (assume dt is 1 day)

$$\sigma_{real}^2 = 252[\ln(S_{dt}/S_0)]^2 = 252[\ln(1+R)]^2 \approx 252R^2$$

$$R^2 = \sigma_{real}^2 dt \quad \Rightarrow \quad P/L = 50\$\Gamma dt(\sigma_{real}^2 - \sigma^2)$$

Delta Hedging v.s. Variance Swap

- Delta hedging and variance swap are similar
  - If realised volatility is higher than implied volatility, delta hedging gains. Otherwise, delta hedging loses money.
  - If realised volatility is higher than the strike, variance swap gains. Otherwise, variance swap loses money.
- Delta hedging and variance swap are different
  - Delta hedging
    - For high dollar gamma (asset price close to strike), the option has high exposure to spread between implied and realised volatilities.
    - For low dollar gamma (asset price far away from the strike), the option has little exposure to volatilities.
    - Such exposure to volatility is path dependent.
  - Variance swap
    - Whatever the price the underlying asset has, variance swap has constant exposure to the spread between realised and strike volatilities.
    - Such exposure to volatility is path independent

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

- Delta Hedging in Black-Scholes
- >>> Replicating a Variance Swap
- The Fair Strike
- Numerical Application

Part III: Variance Swap Strategies

Replicating a Variance Swap

- General idea of replicating variance swap
  - P/L of variance swap: $\text{const} \times (\sigma_{real}^2 - \sigma_{Strike}^2)$
  - P/L of delta hedging: $\text{variable} \times (\sigma_{real}^2 - \sigma_{impl}^2)$
  - Replicating: choose proper weights of options, to achieve constant gamma.
- A mathematical point of view
  - What price function has constant dollar gamma
  - Suppose the price is C(S):
    - Gamma is: $\Gamma = C_{SS}$
    - Dollar gamma is: $\$\Gamma = \Gamma S^2/100 = C_{SS} S^2/100 = \text{constant}$

$$C(S) = -a \ln(S) + bS + c$$

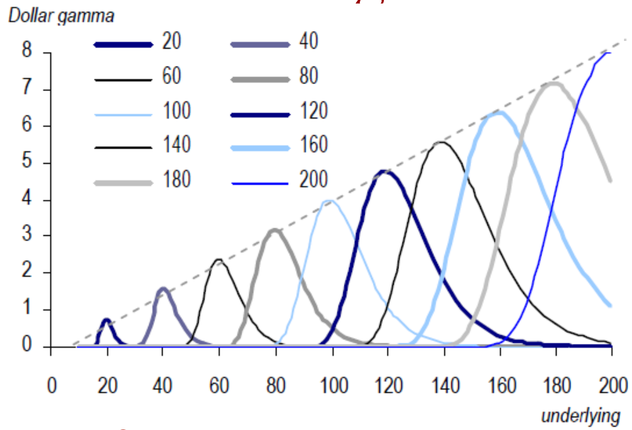
- Payoff of the constant dollar gamma porfolio at T:

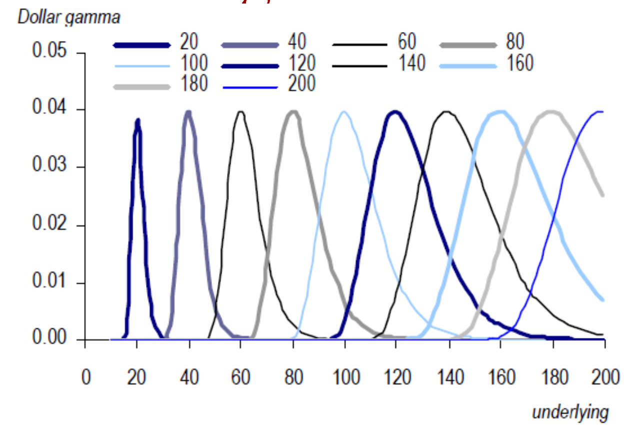
$$-a \ln(S_T) + bS_T + c$$

  - Dollar gamma is: $\$\Gamma = \Gamma S^2/100 = C_{SS} S^2/100 = a$: $a > 0$

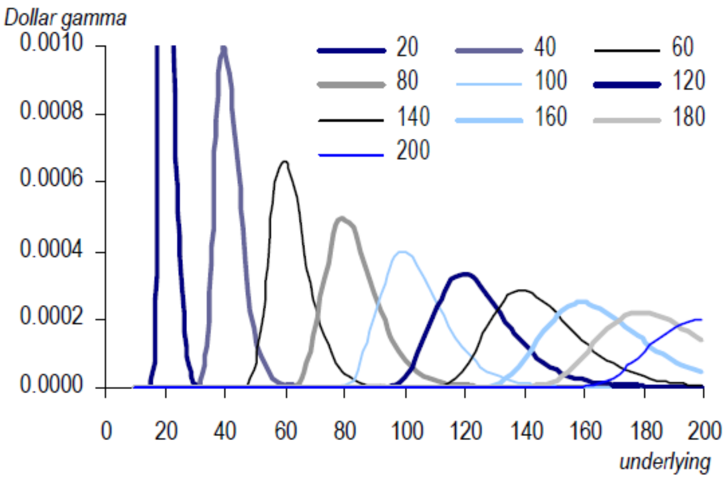
Replicating a Variance Swap

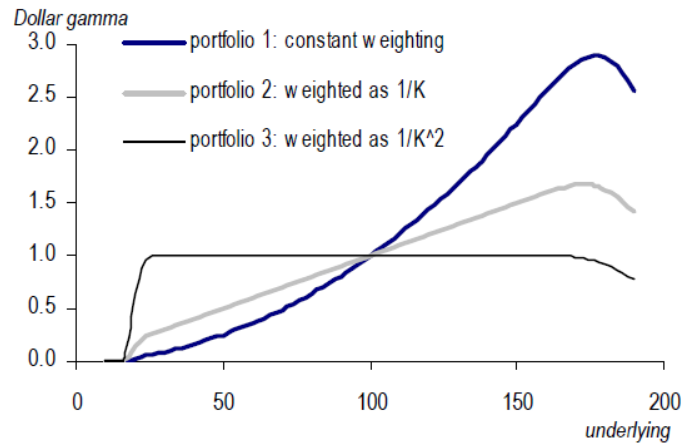
- Constant weighted case v.s. 1/K-weighted case





- $1/K^2$-weighted case & Aggregate dollar gamma





Replicating a Variance Swap

- Use payoffs of vanilla options to construct such payoff:

$$\int_0^{S_0} \frac{(K-S_T)^+}{K^2} dK + \int_{S_0}^{\infty} \frac{(S_T-K)^+}{K^2} dK = \int_{S_0}^{S_T} \frac{S_T-K}{K^2} dK$$
$$= -\ln(S_T) - \frac{1}{S_0} S_T + \ln(S_0) - 1$$

- Conclusion:
  - We can construct a portfolio of calls and puts initially, weighted as 1/ strike-squred, and such portfolio has constant dollar gamma
  - This is a static portfolio, no dynamic trading of options is required.

- It is not commonly used in practice. Why?

Replicating a Variance Swap

- Other possibilities to construct constant dollar gamma portfolio
  - Use a single vanilla option, buy or sell over time, to keep constant dollar gamma.
    - Dynamic trading is needed
    - The position could end up wih enormous amounts of the option.
  - Start with an ATM option, and on each re-hedging step, either sell or hold the previous option, and buy an amound of new ATM to achieve constant dollar gamma
    - Still, dynamic trading of options is needed
  - However, such portfolios are actually used in practice
- Drawbacks of theoretical portforlio
  - Traded strikes are not continuous
  - Lack of liquidity in OTM strikes, especially for puts
- We just use the theoretical model for a fair price

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

- Delta Hedging in Black-Scholes
- Replicating a Variance Swap
- >>> The Fair Strike
- Numerical Application

Part III: Variance Swap Strategies

Derive the Strike

- Ito's formula
  - For arbitrary smooth f(S)

$$f(S_T) - f(S_0) = \int_0^T f'(S_t)dS_t + \int_0^T \frac{1}{2} S_t^2 f''(S_t)\sigma_t^2 dt$$

  - Assume

$$f(S_t) = \frac{2}{T}[\ln(\frac{S_0}{S_t}) + \frac{S_t}{S_0} - 1] \Rightarrow f''(S_t) = \frac{2}{TS_t^2}$$

  - Average realised variance:

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}[\ln(\frac{S_0}{S_T}) + \frac{S_T}{S_0} - 1] - \frac{2}{T}\int_0^T [\frac{1}{S_0} - \frac{1}{S_t}]dS_T$$

$$= \frac{2}{T}\int_0^{S_0} \frac{(K - S_T)^+}{K^2} dK + \frac{2}{T}\int_{S_0}^{\infty} \frac{(S_T - K)^+}{K^2} dK - \frac{2}{T}\int_0^T [\frac{1}{S_0} - \frac{1}{S_t}]dS_T$$

Derive the Strike

- Fair strike:

$$K_{VAR}^2 = \frac{2e^{rT}}{T}\left[\int_0^{S_0} \frac{P_0(K)}{K^2} dK + \int_{S_0}^{\infty} \frac{C_0(K)}{K^2} dK\right]$$

- The last term in realised volatilities
  - It is equivalent as holding $1/S_t - 1/S_0$ units of underlying asset at time t.
  - Initially, no underlying asset is needed.
- Strike calculation in practice
  - Use discrete strike values

$$K_{VAR}^2 = \frac{2e^{rT}}{T}\left[\sum_{K_i \le S_0} \frac{\Delta K_i}{K_i^2} P_0(K_i) + \sum_{K_i > S_0} \frac{\Delta K_i}{K_i^2} C_0(K_i)\right] - \frac{1}{T}\left(\frac{S_0}{K_0} - 1\right)^2$$

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

- Delta Hedging in Black-Scholes
- Replicating a Variance Swap
- The Fair Strike
- >>> Numerical Application

Part III: Variance Swap Strategies

A Real Life Application

- Underlying asset price: $2,935.02
- Chosen strikes: equally apart, every 5% steps
- Calculated strike for variance swap: 16.625%
- Variance notional: 10,000

Floating leg: €2,701,397.53

Fixed leg: €2,701,355.88

Present value: ≈ 0

Weight = $\frac{5\%}{\text{Strike\%}^2}$	Under-lying	Call / Put	Forward	Strike	Strike (%Forward)	Maturity	Implied Volatility	Price (%Notional)
20.00%	SX5E	P	2,935.02	1,467.51	50%	1Y	27.6%	0.04%
16.53%	SX5E	P	2,935.02	1,614.26	55%	1Y	26.4%	0.08%
13.89%	SX5E	P	2,935.02	1,761.01	60%	1Y	25.2%	0.15%
11.83%	SX5E	P	2,935.02	1,907.76	65%	1Y	24.0%	0.27%
10.20%	SX5E	P	2,935.02	2,054.51	70%	1Y	22.7%	0.46%
8.89%	SX5E	P	2,935.02	2,201.26	75%	1Y	21.4%	0.75%
7.81%	SX5E	P	2,935.02	2,348.01	80%	1Y	20.0%	1.17%
6.92%	SX5E	P	2,935.02	2,494.76	85%	1Y	18.7%	1.79%
6.17%	SX5E	P	2,935.02	2,641.51	90%	1Y	17.3%	2.67%
5.54%	SX5E	P	2,935.02	2,788.26	95%	1Y	16.0%	3.94%
2.50%	SX5E	P	2,935.02	2,935.02	100%	1Y	14.8%	5.74%
2.50%	SX5E	C	2,935.02	2,935.02	100%	1Y	14.8%	5.74%
4.54%	SX5E	C	2,935.02	3,081.77	105%	1Y	13.7%	3.37%
4.13%	SX5E	C	2,935.02	3,228.52	110%	1Y	12.9%	1.76%
3.78%	SX5E	C	2,935.02	3,375.27	115%	1Y	12.2%	0.81%
3.47%	SX5E	C	2,935.02	3,522.02	120%	1Y	11.9%	0.35%
3.20%	SX5E	C	2,935.02	3,668.77	125%	1Y	11.8%	0.15%
2.96%	SX5E	C	2,935.02	3,815.52	130%	1Y	11.9%	0.06%
2.74%	SX5E	C	2,935.02	3,962.27	135%	1Y	12.1%	0.03%
2.55%	SX5E	C	2,935.02	4,109.02	140%	1Y	12.5%	0.02%
2.38%	SX5E	C	2,935.02	4,255.77	145%	1Y	12.9%	0.01%
2.22%	SX5E	C	2,935.02	4,402.52	150%	1Y	13.4%	0.01%

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

Part III: Variance Swap Strategies

>>>
- Uses of Variance Swaps
- Rolling Short Variance

Uses of Variance Swaps

- Express local or macro volatility view
  - Buy Swap on Index ahead of recession, sell vol into a spike
  - Better than straddles because not affected by trending underlying, no path dependence, no active management of delta-hedging needed
- Hedging purposes
  - Volatility tends to go up in a slumping market
- Rolling short variance
  - Strategy consists of selling short-dated index variance
  - This is to try and take advantage of the volatility risk premium described earlier
- Diversification
- Index Variance spreads (Vol on S&P 500 vs. EuroStoxx)
- Single-Stock volatility pairs trading
- Correlation and dispersion trading (long index, short basket)
- And many many other….

Outline of Presentation

Part I: Background

Part II: Replicating and Pricing

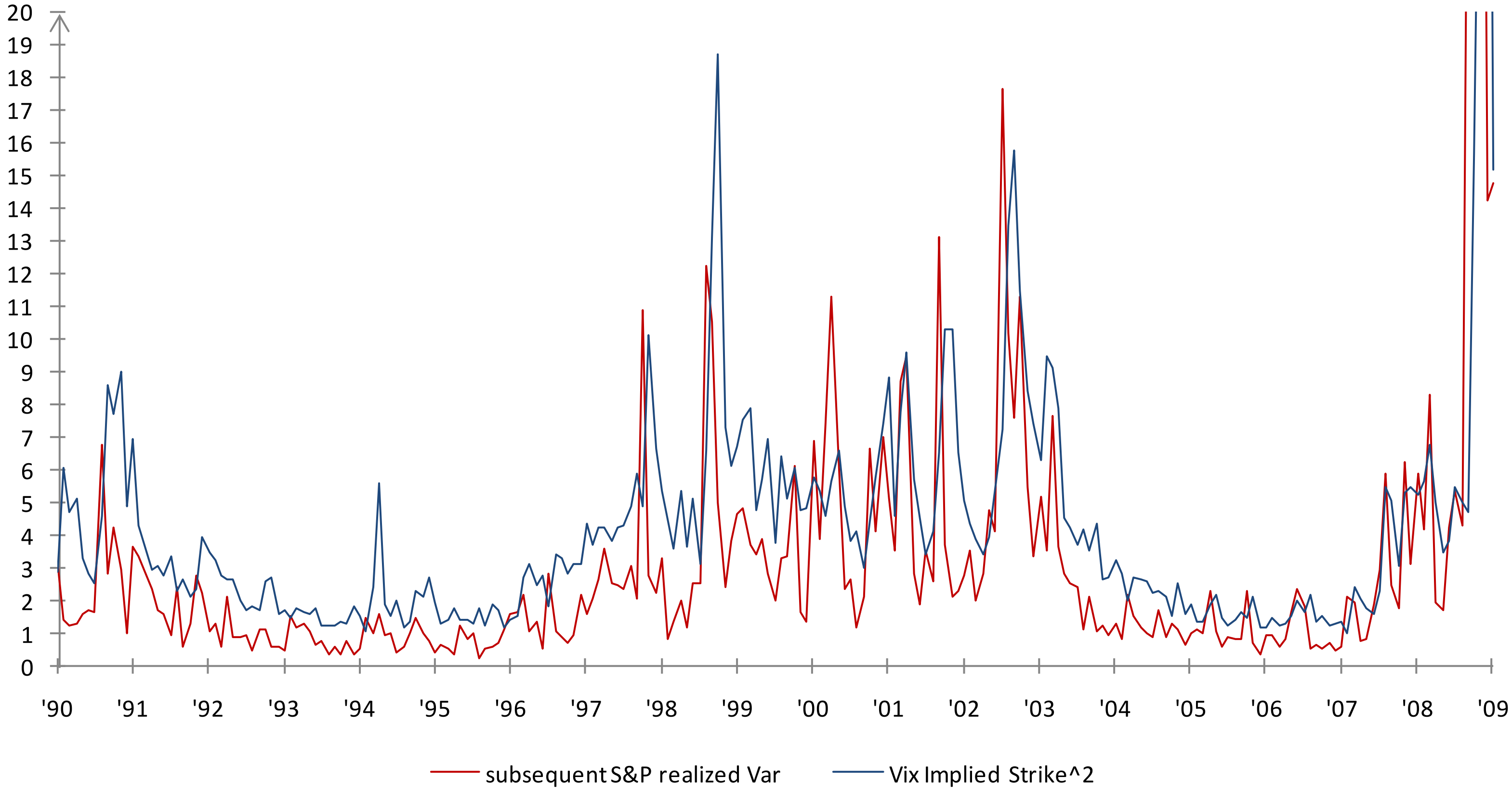
Part III: Variance Swap Strategies

- Uses of Variance Swaps

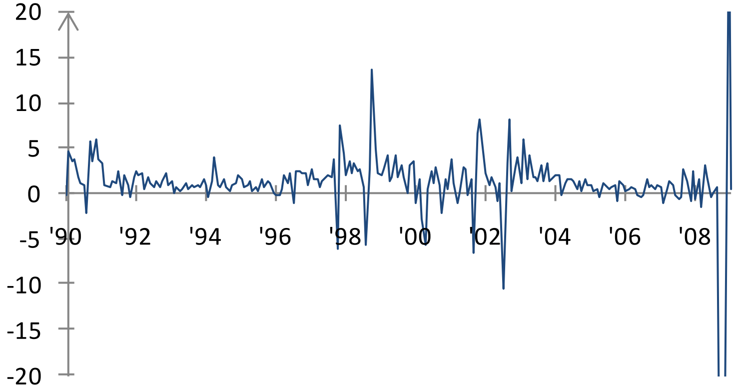
>>> • Rolling Short Variance

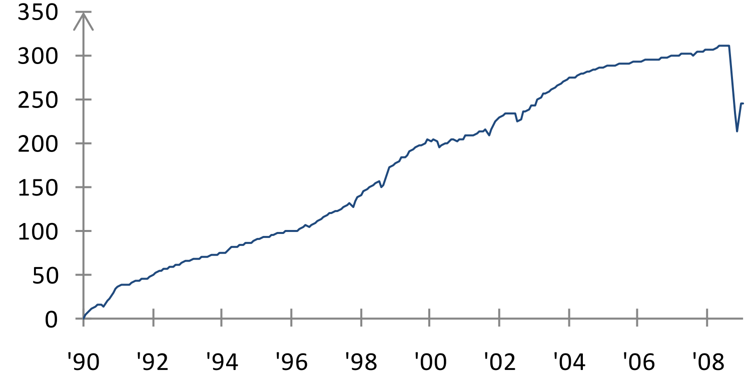
Rolling Short Variance

- Idea: Take advantage of premiums embedded in Vswap
  - Volatility Risk Premium and Convexity Premium
- Short N=$1 Variance Swap at start of each month
- Implied strike taken from VIX in graph below => Actual P&L will be higher



Rolling Short Variance





	'90-'09	'90-'07
Total Months	229	214
% Positive	83.84%	79.91%
% Negative	16.16%	20.09%
Average Return (monthly)	1.09%	1.41%
Largest monthly Return	32.42%	13.73%
Largest monthly Loss	-49.92%	-10.42%
Stdev	5.14%	2.25%

- Note: this example is a conservative estimate
  - Using VIX bid-side instead of actual VS Strikes
  - Not reinvesting gains / accounting losses for tax purposes

End of Presentation

- Questions?