LLT polynomials and Hecke algebra traces

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LLT polynomials

- LLT polynomials are a family of **symmetric functions** introduced by Lascoux, Leclerc and Thibon in 1997 to study Macdonald polynomials.
- **Unicellular** LLT polynomials

$$\mathrm{LLT}_{\mathrm{inc}(P),q} := \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots,$$

where the sum is over arbitrary vertex colorings of $\mathrm{inc}(P)$ for a poset P avoiding $\mathbf{3}+\mathbf{1}$ and $\mathbf{2}+\mathbf{2}$.

- Unicellular LLT polynomials are related to chromatic symmetric functions via a **plethystic substitution**

$$X_{\mathrm{inc}(P),q}[\mathbf{x}] = (q-1)^{-n}\mathrm{LLT}_{\mathrm{inc}(P),q}[(q-1)\mathbf{x}],$$

where n is the size of P.

Hecke algebra $H_n(q)$

Generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by $T_{s_1}, \ldots, T_{s_{n-1}}$ with relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e, && \text{for } i \in [n-1],\\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{for } |i-j| = 1,\\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{for } |i-j| \geq 2. \end{aligned}$$

- $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$.
- Natural basis $\{T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}} \,|\, w = s_{i_1} \cdots s_{i_\ell} \text{ reduced in } \mathfrak{S}_n\}$
- (Modified) Kazhdan-Lusztig basis $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$, where $\widetilde{C}_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v$.

Trace spaces

Let $\mathcal{T}(H_n(q))$ be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module of $H_n(q)$-**traces**, linear functionals $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ with $\theta_q(DD') = \theta_q(D'D)$ for all $D, D' \in H_n(q)$.

- Commonly used bases for $\mathcal{T}(H_n(q))$

$\mathbb{Z}[\mathfrak{S}_n]$	$H_n(q)$
$\{\chi^\lambda \| \lambda \vdash n\}$	$\{\chi_q^\lambda \| \lambda \vdash n\}$
$\{\epsilon^\lambda \| \lambda \vdash n\}$	$\{\epsilon_q^\lambda \| \lambda \vdash n\}$
$\{\eta^\lambda \| \lambda \vdash n\}$	$\{\eta_q^\lambda \| \lambda \vdash n\}$
$\{\psi^\lambda \| \lambda \vdash n\}$	$\{\psi_q^\lambda \| \lambda \vdash n\}$
$\{\phi^\lambda \| \lambda \vdash n\}$	$\{\phi_q^\lambda \| \lambda \vdash n\}$
$\{f^\lambda \| \lambda \vdash n\}$	$\{\gamma_q^\lambda \| \lambda \vdash n\}$

Generating functions

For $D \in \mathbb{Q}(q) \otimes H_n(q)$, we have

$$\begin{aligned} \mathrm{Y}_q(D) &:= \sum_{\lambda \vdash n} \epsilon_q^\lambda(D) m_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(D) f_\lambda \\ &= \sum_{\lambda \vdash n} \frac{\mathrm{sgn}(\lambda)\psi_q^\lambda(D)}{z_\lambda} p_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(D) s_\lambda \\ &= \sum_{\lambda \vdash n} \phi_q^\lambda(D) e_\lambda = \sum_{\lambda \vdash n} \gamma_q^\lambda(D) h_\lambda. \end{aligned}$$

For $w \in \mathfrak{S}_n$ avoiding 312,

$$X_{\mathrm{inc}(P(w)),q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(\widetilde{C}_w(q)) m_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(\widetilde{C}_w(q)) f_\lambda.$$

Plethystic substitution:

$$\mathrm{LLT}_q(D) := (q-1)^n Y_q(D)\left[\frac{1}{q-1}\mathbf{x}\right].$$

Then we have

$$\mathrm{LLT}_q(D) = \sum_{\lambda \vdash n} \epsilon_{q,\mathrm{LLT}}^\lambda(D) m_\lambda = \sum_{\lambda \vdash n} \eta_{q,\mathrm{LLT}}^\lambda(D) f_\lambda.$$

Motivating results

For $w \in \mathfrak{S}_n$ avoiding the pattern 312, we have

$$\epsilon_{q,\mathrm{LLT}}^\lambda(\widetilde{C}_w(q)) = \sum_U q^{\mathrm{INV}_P(U)}, \text{ and}$$

$$\eta_{q,\mathrm{LLT}}^\lambda(\widetilde{C}_w(q)) = \sum_U q^{\mathrm{INV}_P((U_1 \circ \cdots \circ U_r)^R)}.$$

Immanants

- Quantum matrix bialgebra $\mathcal{A}_n(q)$

The noncommutative ring generated as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra by the n^2 variables $t = (t_{i,j})_{i,j \in [n]}$ subject to certain relations.

Given $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n \in \mathfrak{S}_n$, define

$$t^{u,v} := t_{u_1,v_1} \cdots t_{u_n,v_n}.$$

- For any linear functional $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, define the θ_q-immanant in $\mathcal{A}_n(q)$ to be

$$\mathrm{Imm}_{\theta_q}(t) = \sum_{w \in \mathfrak{S}_n} q^{\frac{-\ell(w)}{2}} \theta_q(T_w) t^{e,w}.$$

Open questions

- Let G be a unit interval graph. Find a statistic μ on acyclic orientations of G, such that

$$\mathbf{X}_G(\mathbf{x};q) = \sum_{\theta \in \mathbf{AO}(G)} q^{\mathbf{asc}(\theta)} e_{\mu(\theta)}(\mathbf{x}).$$

- Find a statistic $c_{\mathbf{a}}(T)$ on standard Young tableaux depending on $\mathbf{a}$, such that

$$\mathrm{LLT}_{\mathbf{a}}(\mathbf{x};q) = \sum_{T \in \mathrm{SYT}(n)} q^{c_{\mathbf{a}}(T)} s_{\lambda(T)}.$$

Goals

- Long-term: study unicellular LLT polynomials in the context of Hecke algebra.
- Short-term: describe $\epsilon_{q,\mathrm{LLT}}^\lambda$ and $\eta_{q,\mathrm{LLT}}^\lambda$ in other bases.

Induced sign character

Theorem. *For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we have*

$$\mathrm{Imm}_{\epsilon_{q,\mathrm{LLT}}^\lambda}(t) = \sum_{(I_1,\ldots,I_r)} (t_{I_1,I_1})^{e,e} \cdots (t_{I_r,I_r})^{e,e},$$

where the sum is over all ordered set partitions $(I_1, \ldots, I_r)$ of type λ, and e is the identity permutation in the appropriate subgroup of $\mathfrak{S}_n$.

Corollary. *We have*

$$\epsilon_{q,\mathrm{LLT}}^\lambda = (\epsilon_{q,\mathrm{LLT}}^{\lambda_1} \otimes \cdots \otimes \epsilon_{q,\mathrm{LLT}}^{\lambda_r})\uparrow_{H_\lambda(q)}^{H_n(q)}, \quad \text{where} \quad \epsilon_{q,\mathrm{LLT}}^n(T_w) = \begin{cases} 1 & \text{if } w = e, \\ 0 & \text{otherwise.} \end{cases}$$

Induced trivial character

Theorem. *For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we have*

$$\mathrm{Imm}_{\eta_{q,\mathrm{LLT}}^\lambda}(t) = \sum_{(I_1,\ldots,I_r)} (t_{I_r,I_r})^{w_0,w_0} \cdots (t_{I_1,I_1})^{w_0,w_0},$$

where the sum is over all ordered set partitions of type λ and w_0 is the longest permutation in the appropriate subgroup of $\mathfrak{S}_n$.

Corollary. *We have $\eta_{q,\mathrm{LLT}}^n(T_w) = R_{e,w}(q)$.*

R-polynomials

$\{R_{u,v}(q) \,|\, u, v\}$ are the unique family of polynomials satisfying

1. $R_{u,v}(q) = 0$ if $u \not\leq v$,
2. $R_{v,v}(q) = 1$ for all v, and
3. for each right descent s of v we have

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us < u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{otherwise.} \end{cases}$$