COMBINATORIAL INTERPRETATION OF KAZHDAN–LUSZTIG BASIS ELEMENTS INDEXED BY 45312-AVOIDING PERMUTATIONS IN $\mathfrak{S}_6$

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Abstract. Deodhar [*Geom. Dedicata* **36**, no. 1 (1990)] introduced the *defect* statistic on subexpressions of reduced expressions in the symmetric group $\mathfrak{S}_n$ to construct an algorithmic description of the Kazhdan–Lusztig basis of the Hecke algebra $H_n(q)$. This led Billey–Warrington [*J. Algebraic Combin.* **13**, no. 2 (2001)] and the second author [*J. Pure Appl. Algebra* **212** (2008)] to state very explicit combinatorial descriptions of the basis elements indexed by permutations avoiding certain patterns. We extend the above work to Kazhdan–Lusztig basis elements indexed by $w \in \mathfrak{S}_5, \mathfrak{S}_6$ that avoid the pattern 45312.

1. Introduction

Define the *symmetric group algebra* $\mathbb{Z}[\mathfrak{S}_n]$ and the *(type A Iwahori-) Hecke algebra* $H_n(q)$ to be the algebras with multiplicative identity elements e and T_e, respectively, generated over $\mathbb{Z}$ and $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by elements $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, subject to the relations

$$(1.1)\qquad \begin{array}{lll} s_i^2 = e & T_{s_i}^2 = (q-1)T_{s_i} + qT_e & \text{for } i = 1, \ldots, n-1, \\ s_is_js_i = s_js_is_j & T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j} & \text{for } |i-j| = 1, \\ s_is_j = s_js_i & T_{s_i}T_{s_j} = T_{s_j}T_{s_i} & \text{for } |i-j| \geq 2. \end{array}$$

Analogous to the natural basis $\{w \,|\, w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[\mathfrak{S}_n]$ is the natural basis $\{T_w \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as possible) expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. Specializing at $q^{\frac{1}{2}} = 1$ we have $T_w \mapsto w$ and $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$.

To each element $w = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n$, we associate a *one-line notation* by viewing the generator s_i as a map on words that swaps the letters in positions i and $i+1$, and by defining $w_1 \cdots w_n = s_{i_1}(s_{i_2}(\cdots s_{i_\ell}(12 \cdots n) \cdots))$. For each subinterval $[a, b]$ of $[n] := \{1, \ldots, n\}$, we let $s_{[a,b]}$ denote the element of $\mathfrak{S}_n$ having one-line notation $1 \cdots (a-1)b \cdots a(b+1) \cdots n$, and call such an element a *reversal*. The reversal $s_{[n]}$ is usually denoted w_0. Given a word $a = a_1 \cdots a_k$ in $\mathfrak{S}_k$, and a word $b = b_1 \cdots b_k$ having k distinct letters, we say that b *matches the pattern* a if the letters of b appear in the same relative order as those of a; that is, if we have $a_i < a_j$ if and only if $b_i < b_j$ for all $i, j \in [k]$. Given $w \in \mathfrak{S}_n$ we say that w *avoids the pattern* a if no subword $w_{i_1} \cdots w_{i_k}$ of w matches the pattern a.

A second basis $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$ due to Kahzdan and Lusztig [7] expands in the natural basis as

$$(1.2)\qquad \widetilde{C}_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v,$$

Date: March 6, 2022.

where $\leq$ is the Bruhat order, and where the coefficients $P_{v,w}(q)$ belong to $\mathbb{N}[q]$ and are called *Kazhdan–Lusztig polynomials.* While the Kazhdan–Lusztig basis is important in various areas of mathematics, we don't have a very simple description of it or of the polynomials which relate it to the natural basis of $H_n(q)$. On the other hand, when $w \in \mathfrak{S}_n$ avoids certain patterns, we can factor $\widetilde{C}_w(q)$ as a product of simpler Kazhdan–Lusztig basis elements indexed by reversals. Such a product then produces a directed graph called a *planar network*, which in turn provides combinatorial interpretations of the coefficients in each polynomial $P_{v,w}(q)$ for $v \leq w$.

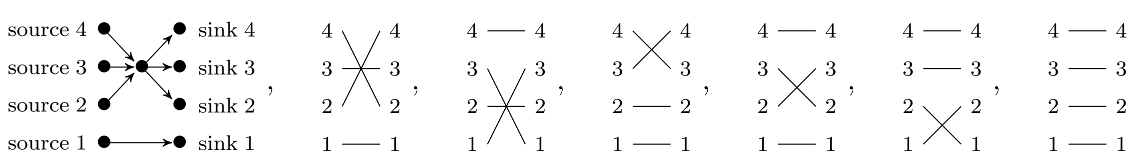
In Section 2 we review planar networks and classes of these used by Billey–Warrington and the second author to represent certain Kazhdan–Lusztig basis elements. In Section 3 we present our main results which suggest a common generalization of the results in the previous section.

2. Planar networks and graphical representation of elements of $H_n(q)$

Define a *planar network of order* n to be a directed, planar, acyclic graph which can be embedded in a disc so that $2n$ boundary vertices can be labeled counterclockwise as *source* $1, \ldots,$ *source* n, *sink* $n, \ldots,$ *sink* 1. We will assume that all sources have indegree 0 and all sinks have outdegree 0. Let $\mathcal{G}_n$ denote the set of such networks. For each subinterval $[a, b]$ of $[n]$ we define a *simple star network* $G_{[a,b]} \in \mathcal{G}_n$ by

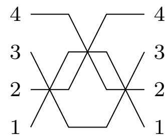
(1) An interior vertex z lies between the sources and sinks.
(2) For $i \in [a, b]$ we have directed edges (source i, z) and (z, sink i).
(3) For $i \notin [a, b]$ we have directed edges (source i, sink i).

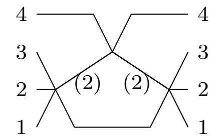
For zero- and one-element subintervals we define the trivial network $G_\emptyset = G_{[1,1]} = \cdots = G_{[n,n]}$ to have no interior vertex, and n edges, each from source i to sink i, for $i = 1, \ldots, n$. In figures, we will draw sources on the left and sinks on the right, both numbered from bottom to top. To economize figures, we will omit vertices and edge orientations (always left to right). The (infinite) set $\mathcal{G}_4$ contains seven simple star networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_\emptyset = G_{[1,1]} = \cdots = G_{[4,4]}$, respectively,

(2.1)


where we have drawn $G_{[1,4]}$ in full detail and other networks in economical form.

We write $G \circ H$ for concatenation of G and H, formed by identifying sink i of G with source i of H to form an internal vertex, for $i = 1, \ldots, n$. The sources of $G \circ H$ are those of G, and the sinks of $G \circ H$ are those of H. Sometimes a concatenation $G \circ H$ may be a multi-digraph, because for some vertices $x \in G$, $y \in H$, a collection of $m(x, y) > 1$ edges are incident upon both. Define $G \bullet H$ to be the simple subgraph of $G \circ H$ obtained by removing, for all such pairs (x, y), all but one of the $m(x, y)$ edges incident upon both, and by marking this edge with the multiplicity $m(x,y)$. For example, in $\mathcal{G}_4$ we have the nonisomorphic graphs

(2.2) $G_{[1,3]} \circ G_{[2,4]} \circ G_{[1,3]} =$ 

, $\qquad G_{[1,3]} \bullet G_{[2,4]} \bullet G_{[1,3]} =$ 

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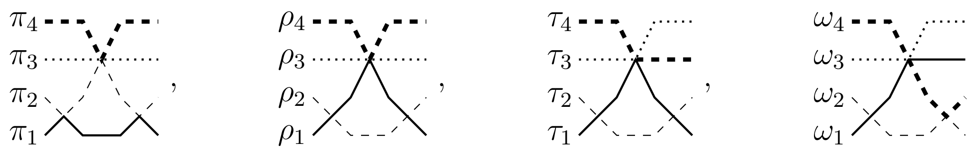
in which two pairs of edges are replaced by two single edges marked with multiplicity 2. Define a *star network* to be the concatenation of finitely many simple star networks, using any combination of the $\circ$ and $\bullet$ operations.

The graphical representation of $H_n(q)$-elements depends upon families of paths in star networks, and upon a function on these paths called the *defect statistic*. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a sequence of source-to-sink paths in a star network G. We call π a *path family* if there exists a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ such that π_i is a path from source i to sink w_i. In this case, we say more specifically that π has *type* w. We say that the path family *covers* G if it contains every edge with exactly the multiplicity of that edge. For example, the stars in the star network $G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$ imply that there are $2 \cdot 6 \cdot 2 = 24$ path families that cover it. Four of these are

(2.3) 

type$(\pi) = 1234$ $\qquad$ type$(\rho) = 1234$ $\qquad$ type$(\tau) = 1243$ $\qquad$ type$(\omega) = 3142$

Suppose that path family π covers a star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$. If two paths π_i, π_j intersect at the central vertex of G_{J_p}, call the triple (π_i, π_j, p) *defective* or *a defect* if the paths have previously crossed an odd number of times (i.e., in $G_{J_1}, \ldots, G_{J_{i_{p-1}}}$). Let $\mathrm{dfct}(\pi)$ denote the number of defects of π,

$$\mathrm{dfct}(\pi) = \#\{(\pi_i, \pi_j, p) \,|\, (\pi_i, \pi_j, p) \text{ defective}\}. \tag{2.4}$$

For example, in (2.3) we have $\mathrm{dfct}(\rho) = \mathrm{dfct}(\tau) = 1$, since ρ_1, ρ_2 cross and meet again later, as do τ_1, τ_2, and we have $\mathrm{dfct}(\pi) = \mathrm{dfct}(\omega) = 0$.

To a planar network G we associate an $H_n(q)$ element

$$\beta_q(G) = \sum_{\pi} q^{\mathrm{dfct}(\pi)} T_{\mathrm{type}(\pi)}, \tag{2.5}$$

where the sum is over all path families that cover G, and we say that G *graphically represents* $\beta_q(G)$, i.e., G gives an explicit expansion of $\beta_q(G)$ in the natural basis of $H_n(q)$. Deodhar [4] showed that for each expression $s_{i_1} \cdots s_{i_m}$, the *wiring diagram* $G_{[i_1,i_1+1]} \circ \cdots \circ G_{[i_m,i_m+1]}$ satisfies $\beta_q(G_{[i_1,i_1+1]} \circ \cdots \circ G_{[i_m,i_m+1]}) = \widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q)$. Billey–Warrington [1, Thm. 1] showed that for some $w \in \mathfrak{S}_n$, each reduced expression $s_{i_1} \cdots s_{i_\ell}$ for w satisfies $\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q) = \widetilde{C}_w(q)$. This implies the following graphical representation result.

Theorem 2.1. *Let $w \in \mathfrak{S}_n$ avoid the patterns* 321, 56781234, 56718234, 46781235, 46718235, *and let G be the wiring diagram for any reduced expression for w. Then G graphically represents $\widetilde{C}_w(q)$.*

We also have the following generalization of Deodhar's result [3, Cor. 5.3].

Theorem 2.2. *For each sequence $(s_{[a_1,b_1]},\dots,s_{[a_t,b_t]})$ of reversals, we have*

$$\beta_q(G_{[a_1,b_1]}\circ\cdots\circ G_{[a_1,b_1]})=\widetilde{C}_{s_{[a_1,b_1]}}(q)\cdots\widetilde{C}_{s_{[a_t,b_t]}}(q).$$

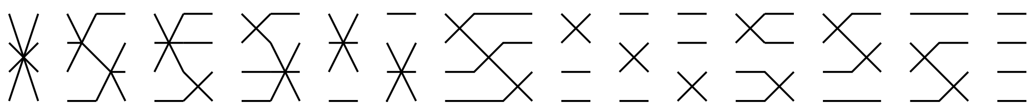
Some Kazhdan-Lusztig basis elements not included in Theorem 2.1 have simple graphical representations which are generalizations of wiring diagrams. Call a star network of the form

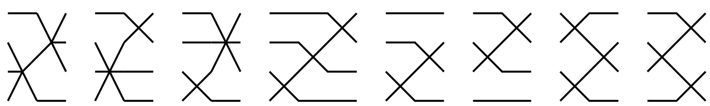
$$G=G_{[c_1,d_1]}\bullet\cdots\bullet G_{[c_t,d_t]} \tag{2.6}$$

a *zig-zag network* if

1. the sequence $([c_1,d_1],\dots,[c_t,d_t])$ consists of t distinct, pairwise nonnesting intervals,
2. for $i<j<k$, if $[c_i,d_i]\cap[c_j,d_j]\neq\emptyset$ and $[c_j,d_j]\cap[c_k,d_k]\neq\emptyset$, then we have $c_i<c_j<c_k$ (and $d_i<d_j<d_k$) or $c_i>c_j>c_k$ (and $d_i>d_j>d_k$).

The zig-zag networks of order 4 are

(2.7)  ,

(2.8)  .

It was shown in [9, Thm. 3.5, Lem. 5.3] that zig-zag networks of order n correspond bijectively to 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_n$. Letting $G(w)$ be the zig-zag network corresponding to w, we have the following by [9, Thm. 4.3] and Theorem 2.2.

Theorem 2.3. *Let $w\in\mathfrak{S}_n$ avoid the patterns 3412 and 4231. Then the zig-zag network $G(w)$ graphically represents $\widetilde{C}_w(q)$.*

For example, the second zig-zag network in (2.7) is a graphical representation of $\widetilde{C}_{3421}(q)$. Applying Theorem 2.3 and Theorem 2.2 to the third zig-zag network in (2.7), we have that it is a graphical representation of $\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q)=\widetilde{C}_{2431}(q)$. Applying Theorem 2.2 to the star network in (2.3), we find that it is a graphical representation of $\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q)$. This product is precisely $\widetilde{C}_{4231}(q)$, although the equality is not implied by Theorem 2.1 or 2.3. This raises the following question.

Question 2.4. *For which $w\in\mathfrak{S}_n$ is there a star network G satisfying $\beta_q(G)=\widetilde{C}_w(q)$?*

3. New results

A star network G satisfying $\beta_q(G)=\widetilde{C}_w(q)$ can provide graphical representations for Kazhdan–Lusztig basis elements $\widetilde{C}_v(q)$ for v related to w. Let G^R and G^U be the star networks obtained by reflecting G in a vertical line, and horizontal line, respectively, and let $G^{RU}=G^{UR}$ be the result of performing both reflections.

Lemma 3.1. *Fix $w\in\mathfrak{S}_n$ and let star network G satisfy $\beta_q(G)=\widetilde{C}_w(q)$. Then we have $\beta_q(G^R)=\widetilde{C}_{w^{-1}}(q)$, $\beta_q(G^U)=\widetilde{C}_{w_0ww_0}(q)$, and $\beta_q(G^{UR})=\widetilde{C}_{w_0w^{-1}w_0}(q)$.*

For example, let G be the third network in (2.7), which graphically represents $\widetilde{C}_{2431}(q)$, and define $w=2431$. Related to w are $w_0ww_0=4213$, $w^{-1}=4132$, $w_0w^{-1}w_0=3241$. The corresponding Kazhdan–Lusztig basis elements $\widetilde{C}_{4213}(q)$, $\widetilde{C}_{4132}(q)$, $\widetilde{C}_{3241}(q)$ are graphically represented by G^U, G^R, G^{UR}, which appear second and third in (2.8), and fourth in (2.7).

We now answer the special case $n=5$ of Question 2.4.

Theorem 3.2. *For all $w \in \mathfrak{S}_5 \smallsetminus \{45312\}$, there is a star network G satisfying $\beta_q(G) = \widetilde{C}_w(q)$.*

Proof. (Idea) By Theorems 2.1 and 2.3, we have a network G for all permutation avoiding the patterns listed in those theorems. Partitioning the remaining elements of $\mathfrak{S}_5$ into equivalence classes of the form $w \sim w^{-1} \sim w_0 w w_0 \sim w_0 w^{-1} w_0$, we find a zig-zag network $G(w)$ for one representative of each class except for the singleton class $\{45312\}$. Lemma 3.1 gives zig-zag networks for the other elements of each class. □

We prove that in $\mathfrak{S}_6$, an analog of Theorem 3.2 in Theorem 3.3.

Theorem 3.3. *For all $w \in \mathfrak{S}_6$ avoiding the pattern 45312, there is a star network G satisfying $\beta_q(G) = \widetilde{C}_w(q)$.*

Proof. By Theorems 2.1 and 2.3, we have a network G for all permutations avoiding the patterns listed in those theorems. Now from the remaining elements of $\mathfrak{S}_6$, restrict attention to those avoiding the pattern 45312 and partition these into equivalence classes of the form $w \sim w^{-1} \sim w_0 w w_0 \sim w_0 w^{-1} w_0$. We consider three (?) cases of such equivalence classes. If the equivalence class contains an element w which satisfies X, then apply Lemma **??**. If the equivalence class contains an element w which satisfies Y, then apply Lemma **??**. Finally, if the equivalence class contains an element w which satisfies Z, then apply Lemma **??**. □

Problem 3.4. *Given $w \in \mathfrak{S}_n$ such that $\widetilde{C}_w(q)$ can be graphically represented by a star network, explain algorithmically how to produce one such network.*

Is it true that
$\star$ Continue here and include some material from the following (temporary) section.

4. Products of Kazhdan–Lusztig basis elements

Fix $w \in \mathfrak{S}_n$ and an adjacent transposition s. It is known that if $sw > w$ in the Bruhat order then we have

$$\widetilde{C}_s(q)\widetilde{C}_w(q) = \widetilde{C}_{sw}(q) + \sum_{\substack{v<w\\ sv<v}} \mu(v,w)\widetilde{C}_v(q); \tag{4.1}$$

if $ws > w$ then we have

$$\widetilde{C}_w(q)\widetilde{C}_s(q) = \widetilde{C}_{ws}(q) + \sum_{\substack{v<w\\ vs<v}} \mu(v,w)\widetilde{C}_v(q). \tag{4.2}$$

(See, e.g., [5, Appendix].)

($\star$ Include other parts of the formula: $\widetilde{C}_s(q)\widetilde{C}_w(q) = (1+q)\widetilde{C}_w(q)$ if $sw < w$, etc.)

Lemma 4.1. *Fix $w \in \mathfrak{S}_6$ avoiding the patterns 3412 and 4231 and an adjacent transposition s. If we have $sw > w$ and no permutation $v < w$ satisfies $sv < v$ and $\ell(v) = \ell(w) - 1$, then $\widetilde{C}_{sw}(q) = \widetilde{C}_s(q)\widetilde{C}_w(q)$. Equivalently, if we have $ws > w$ and no permutation $v < w$ satisfies $vs < v$ and $\ell(v) = \ell(w) - 1$, then $\widetilde{C}_{ws}(q) = \widetilde{C}_w(q)\widetilde{C}_s(q)$.*

Proof. Recall that $\mu(v,w)$ is the coefficient of $q^{(\ell(w)-\ell(v)-1)/2}$ in $P_{v,w}(q)$. Since w avoids the patterns 3412 and 4231, we have that $P_{v,w}(q) = 1$ for all $v \leq w$. Thus this coefficient can be nonzero only when $(\ell(w) - \ell(v) - 1)/2 = 0$, i.e., when $\ell(v) = \ell(w) - 1$. It follows that if no such permutation v satisfies $sv < v$, then (4.1) reduces to $\widetilde{C}_s(q)\widetilde{C}_w(q) = \widetilde{C}_{sw}(q)$. The second claimed equality is proved similarly. □

Corollary 4.2. *We have* $\widetilde{C}_{52431}(q) = \widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,5]}}(q)\widetilde{C}_{s_{[1,2]}}(q)$.

Proof. Observe that we have

$$\widetilde{C}_{s_{[2,5]}}(q)\widetilde{C}_{s_{[1,2]}}(q) = \widetilde{C}_{25431}(q). \tag{4.3}$$

and that 25431 avoids the patterns 3412 and 4231. Applying Lemma 4.1, write

$$w = 25431 < s_1 w = 52431,$$

and look for v with $\ell(v) = \ell(25431) - 1$ and $s_1 v < v < 25431$. That is, v must satisfy

(i) $v_1 > v_2$ (since we want $s_1 v < v$),
(ii) v is obtainable from $w = 25431$ by swapping letters $w_i > w_j$ in positions $i < j$ (since we want $v < w$) such that no letters $w_{i+1}, \ldots, w_{j-1}$ have values between w_i and w_j (since we want $\ell(v) = \ell(w) - 1$).

But there is no such permutation v: the only swap of a larger letter with a later smaller letter which gives us $v_1 > v_2$ is the swap of 5 with 1. But this decreases inversions by more than 1 because of the 4 and 3 located between the 5 and 1. Thus by Lemma 4.1 we have

$$\widetilde{C}_{52431}(q) = \widetilde{C}_{s_1}(q)\widetilde{C}_{25431}(q),$$

and by (4.3) we have the desired factorization. □

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