BARRETT-JOHNSON INEQUALITIES FOR TOTALLY NONNEGATIVE MATRICES

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Outline

(1) HPSD and TNN matrices
(2) Classical inequalities
(3) The Barrett-Johnson inequality for real PSD matrices
(4) Extension to TNN matrices
(5) Open problems

HPSD and TNN matrices

Given $n \times n$ matrix $A = (a_{i,j})$, subsets $I, J \subset [n] := \{1, \ldots, n\}$, define submatrix $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$.

Call A *Hermitian* (H) if $A^* = A$,
positive semidefinite (PSD) if $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$,
where * denotes conjugate transpose.

Call A *totally nonnegative* (TNN) if
$\det(A_{I,J}) \geq 0$ for all $I, J \subseteq [n]$, $|I| = |J|$.

Fact: A is HPSD if $a^*_{j,i} = a_{i,j}$ for all $i, j \in [n]$,
$\det(A_{I,I}) \geq 0$ for all $I \subseteq [n]$.

Classical inequalities for A HPSD or TNN

Hadamard (HPSD), Koteljanskii (TNN):

$$\det(A) \leq a_{1,1} \cdots a_{n,n}.$$

Fischer (HPSD), Fan (TNN): For all $I \subseteq [n]$, $\overline{I} := [n] \smallsetminus I$,

$$\det(A) \leq \det(A_{I,I}) \det(A_{\overline{I},\overline{I}}).$$

Schur (HPSD), Stembridge (TNN): For all $\mathfrak{S}_n$-characters χ,

$$\det(A) \leq \frac{\mathrm{Imm}_\chi(A)}{\chi(e)},$$

where

$$\mathrm{Imm}_\chi(A) := \sum_{w \in \mathfrak{S}_n} \chi(w) a_{1,w(1)} \cdots a_{1,w(n)}.$$

Barrett-Johnson inequalities for A real PSD

Given $\lambda = (\lambda_1, \ldots, \lambda_r), \mu = (\mu_1, \ldots, \mu_s) \vdash n$, we have

$$\sum_{\substack{(I_1,\ldots,I_r) \\ |I_i|=\lambda_i \\ I_i \cap I_j = \emptyset}} \frac{\det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r})}{\binom{n}{\lambda_1,\ldots,\lambda_r}} \leq \sum_{\substack{(J_1,\ldots,J_s) \\ |J_i|=\mu_i \\ I_i \cap I_j = \emptyset}} \frac{\det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s})}{\binom{n}{\mu_1,\ldots,\mu_s}},$$

for all A real PSD if and only if $\lambda \succeq \mu$ in dominance.

Sums are over *ordered set partitions of* $[n]$ of type λ and μ; the numbers of these are

$$\binom{n}{\lambda_1, \ldots, \lambda_r} = \frac{n!}{\lambda_1! \cdots \lambda_r!}, \qquad \binom{n}{\mu_1, \ldots, \mu_s} = \frac{n!}{\mu_1! \cdots \mu_s!}.$$

Main result

Theorem: (SS '22) The Barrett-Johnson inequalities hold for TNN matrices.

Example: For $n = 4$ we have $4 \succeq 31 \succeq 22 \succeq 211 \succeq 1111$ and

$$\begin{aligned}
\det(A) &\leq \frac{\det(A_{123,123})a_{4,4} + \cdots + \det(A_{234,234})a_{1,1}}{4} \\
&\leq \frac{\det(A_{12,12})\det(A_{34,34}) + \cdots + \det(A_{34,34})\det(A_{12,12})}{6} \\
&\leq \frac{\det(A_{12,12})a_{3,3}a_{4,4} + \cdots + \det(A_{34,34})a_{2,2}a_{1,1}}{12} \\
&\leq \frac{a_{1,1}a_{2,2}a_{3,3}a_{4,4} + \cdots + a_{4,4}a_{3,3}a_{2,2}a_{1,1}}{24}.
\end{aligned}$$

$$
\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 1&1&1&1\\\hline\end{array}
$$

$$
\leq \frac{1}{4}\left(
\begin{array}{|c|c|c|c|}\hline 1&1&1&\\\hline 1&1&1&\\\hline 1&1&1&\\\hline &&&2\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&1&&1\\\hline 1&1&&1\\\hline &&2&\\\hline 1&1&&1\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&1&1\\\hline &2&&\\\hline 1&&1&1\\\hline 1&&1&1\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 2&&&\\\hline &1&1&1\\\hline &1&1&1\\\hline &1&1&1\\\hline\end{array}
\right)
$$

$$
\leq \frac{1}{6}\left(
\begin{array}{|c|c|c|c|}\hline 1&1&&\\\hline 1&1&&\\\hline &&2&2\\\hline &&2&2\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&1&\\\hline &2&&2\\\hline 1&&1&\\\hline &2&&2\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&&1\\\hline &2&2&\\\hline &2&2&\\\hline 1&&&1\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 2&&&2\\\hline &1&1&\\\hline &1&1&\\\hline 2&&&2\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 2&&2&\\\hline &1&&1\\\hline 2&&2&\\\hline &1&&1\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 2&2&&\\\hline 2&2&&\\\hline &&1&1\\\hline &&1&1\\\hline\end{array}
\right)
$$

$$
\leq \frac{1}{12}\left(
\begin{array}{|c|c|c|c|}\hline 1&1&&\\\hline 1&1&&\\\hline &&2&\\\hline &&&3\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&1&&\\\hline 1&1&&\\\hline &&3&\\\hline &&&2\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&1&\\\hline &2&&\\\hline 1&&1&\\\hline &&&3\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&1&\\\hline &3&&\\\hline 1&&1&\\\hline &&&2\\\hline\end{array}
+\cdots+
\begin{array}{|c|c|c|c|}\hline 3&&&\\\hline &2&&\\\hline &&1&1\\\hline &&1&1\\\hline\end{array}
\right)
$$

$$
\leq \frac{1}{24}\left(
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &2&&\\\hline &&3&\\\hline &&&4\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &2&&\\\hline &&4&\\\hline &&&3\\\hline\end{array}
+
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &3&&\\\hline &&2&\\\hline &&&4\\\hline\end{array}
+\cdots+
\begin{array}{|c|c|c|c|}\hline 4&&&\\\hline &3&&\\\hline &&2&\\\hline &&&1\\\hline\end{array}
\right)
=
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &2&&\\\hline &&3&\\\hline &&&4\\\hline\end{array}\,.
$$

Connection to induced sign characters

Define the *induced trivial* and *induced sign* characters by

$$\eta^\lambda = \mathrm{triv}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \qquad \epsilon^\lambda = \mathrm{sgn}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}.$$

Theorem: (Littlewood '40, Merris–Watkins '85)

$$\mathrm{Imm}_{\epsilon^\lambda}(A) = \sum_{(I_1,\ldots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}),$$

$$\mathrm{Imm}_{\eta^\lambda}(A) = \sum_{(I_1,\ldots,I_r)} \mathrm{per}(A_{I_1,I_1}) \cdots \mathrm{per}(A_{I_r,I_r}),$$

summed over ordered set partitions of type $\lambda = (\lambda_1, \ldots, \lambda_r)$.

Barrett-Johnson inequalities can be rewritten as

$$\frac{1}{\binom{n}{\lambda_1,\ldots,\lambda_r}} \mathrm{Imm}_{\epsilon^\lambda}(A) \leq \frac{1}{\binom{n}{\mu_1,\ldots,\mu_s}} \mathrm{Imm}_{\epsilon^\mu}(A).$$

Proof idea

It suffices to show that for all TNN A and $k \leq \lfloor \frac{n}{2} \rfloor - 1$ we have

$$\sum_{|I|=k} \frac{\det(A_{I,I})\det(A_{\overline{I},\overline{I}})}{\binom{n}{k}} \leq \sum_{|I|=k+1} \frac{\det(A_{I,I})\det(A_{\overline{I},\overline{I}})}{\binom{n}{k+1}},$$

equivalently,

$$\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1}} - \frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k}} \geq 0. \qquad (1)$$

We prove (1) in two ways:

1. using monomial trace immanants,
2. using Temperley-Lieb immanants.

Monomial trace version of proof

Let $\{\phi^\lambda \mid \lambda \vdash n\}$ be the *monomial traces* of $\mathfrak{S}_n$,

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu, \qquad \chi^\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} \phi^\mu.$$

Conjecture: (Stembridge, '91) For all $n \times n$ TNN matrices A and all partitions $\lambda \vdash n$ we have $\mathrm{Imm}_{\phi^\lambda}(A) \geq 0$.

Theorem: (CHSS, '16) For all $n \times n$ TNN matrices A and all partitions $\lambda \vdash n$ with $\lambda_1 \leq 2$ we have $\mathrm{Imm}_{\phi^\lambda}(A) \geq 0$.

Proposition: There are $\{c_{k,\mu} \mid \mu \vdash n, \mu_1 \leq 2\} \subseteq \mathbb{N}$ satisfying

$$\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1}} - \frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k}} = \sum_{\substack{\mu \vdash n \\ \mu_1 \leq 2}} c_{k,\mu} \mathrm{Imm}_{\phi^\mu}(A).$$

Temperley-Lieb version of proof

The Temperley-Lieb algebra $T_n(2)$ is generated over $\mathbb{C}$ by $t_1, \ldots, t_{n-1}$ subject to relations

$$\begin{aligned} t_i^2 &= 2t_i && i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i && \text{if } |i-j| \geq 2. \end{aligned}$$

Using $\mathfrak{S}_n$-generators $s_1, \ldots, s_{n-1}$ define the map

$$\begin{aligned} \sigma : \mathbb{C}[\mathfrak{S}_n] &\to T_n(2), \\ s_i &\mapsto t_i - 1. \end{aligned}$$

This is surjective with kernel equal to the ideal

$$(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1),$$

and image spanned by the *standard basis* of $T_n(2)$

$$\mathcal{B}_n = \{\sigma(w) \,|\, w \in \mathfrak{S}_n \text{ avoids the pattern } 321\}.$$

For each $\tau \in \mathcal{B}_n$, define a linear functional

$$f_\tau : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{R},$$
$$w \mapsto \text{coefficient of } \tau \text{ in } \sigma(w).$$

and the immanant $\mathrm{Imm}_\tau(A) := \mathrm{Imm}_{f_\tau}(A)$.

Theorem: (RS, '05) For all $n \times n$ TNN matrices A and all $\tau \in \mathcal{B}_n$ we have $\mathrm{Imm}_\tau(A) \geq 0$.

Proposition: There are $\{d_{k,\tau} \mid \tau \in \mathcal{B}_n\} \subseteq \mathbb{N}$ satisfying

$$\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1}} - \frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k}} = \sum_{\tau \in \mathcal{B}_n} d_{k,\tau} \mathrm{Imm}_\tau(A).$$

More classical inequalities for A HPSD or TNN

Marcus (HPSD), clear (TNN):

$$\mathrm{per}(A) \geq a_{1,1} \cdots a_{n,n}.$$

Lieb (HPSD), clear (TNN): For all $I \subseteq [n]$,

$$\mathrm{per}(A) \geq \mathrm{per}(A_{I,I})\mathrm{per}(A_{\overline{I},\overline{I}}).$$

Conj. by Lieb (HPSD), Johnson (TNN): For all $\mathfrak{S}_n$-characters χ,

$$\mathrm{per}(A) \geq \frac{\mathrm{Imm}_{\chi}(A)}{\chi(e)}.$$

Open problems

Problem: Show that the Barrett-Johnson inequalities hold for all HPSD matrices.

Problem: Characterize the pairs (λ, μ) of partitions of n for which a permanental analog of the Barrett-Johnson inequalities

$$\frac{1}{\binom{n}{\lambda_1,\ldots,\lambda_r}}\mathrm{Imm}_{\eta^\lambda}(A) \geq \frac{1}{\binom{n}{\mu_1,\ldots,\mu_s}}\mathrm{Imm}_{\eta^\mu}(A). \tag{2}$$

holds for all HPSD or real PSD matrices.

Problem: Characterize the pairs (λ, μ) of partitions of n for which (2) holds for all TNN matrices.