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Abstract

We define *Dumont's statistic* on the symmetric group S_n to be the function dmc: $S_n \to \mathbb{N}$ which maps a permutation σ to the number of distinct nonzero letters in code(σ). Dumont showed that this statistic is Eulerian [6]. Naturally extending Dumont's statistic to the rearrangement classes of arbitrary words, we create a generalized statistic which is again Eulerian. As a consequence, we show that for each distributive lattice $J(P)$ which is a product of chains, there is a poset Q such that the f-vector of Q is the h-vector of $J(P)$. This strengthens for products of chains a result of Stanley concerning the flag h-vectors of Cohen-Macaulay complexes [9, Cor. 4.5]. We conjecture that the result holds for all finite distributive lattices.

Dumont's statistic on words

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August 17, 2000

1 Introduction

Let S_n be the symmetric group on n letters, and let us write each permutation π in S_n in one line notation: $\pi = \pi_1 \cdots \pi_n$. We call position i a *descent* in π if $\pi_i > \pi_{i+1}$, and an *excedance* in π if $\pi_i > i$. Counting descents and excedances, we define two permutation statistics $\mathrm{des} : S_n \to \mathbb{N}$ and $\mathrm{exc} : S_n \to \mathbb{N}$ by

$$\mathrm{des}(\pi) = \#\{i \,|\, \pi_i > \pi_{i+1}\},$$
$$\mathrm{exc}(\pi) = \#\{i \,|\, \pi_i > i\}.$$

It is well known that the number of permutations in S_n with k descents equals the number of permutations in S_n with k excedances. This number is often denoted $A(n, k+1)$ and the generating function

$$A_n(x) = \sum_{k=0}^{n-1} A(n, k+1)x^{k+1} = \sum_{\pi \in S_n} x^{1+\mathrm{des}(\pi)} = \sum_{\pi \in S_n} x^{1+\mathrm{exc}(\pi)}$$

is called the nth *Eulerian polynomial.* Any permutation statistic $\mathrm{stat} : S_n \to \mathbb{N}$ satisfying

$$A_n(x) = \sum_{\pi \in S_n} x^{1+\mathrm{stat}(\pi)},$$

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or equivalently,

$$\#\{\pi \in S_n \mid \text{stat}(\pi) = k\} = \#\{\pi \in S_n \mid \text{des}(\pi) = k\}, \text{for } k = 0, \ldots, n-1$$

is called *Eulerian*.

A third Eulerian statistic, essentially defined by Dumont [6], counts the number of distinct nonzero letters in the code of a permutation. We define code(π) to be the word $c_1 \cdots c_n$, where

$$c_i = \#\{j > i \mid \pi_j < \pi_i\}.$$

Denoting Dumont's statistic by *dmc*, we have

$$\text{dmc}(\pi) = \#\{\ell \neq 0 \mid \ell \text{ appears in code}(\pi)\}.$$

Example 1.1.

$$\pi = 2\ 8\ 4\ 3\ 6\ 7\ 9\ 5\ 1,$$
$$\text{code}(\pi) = 1\ 6\ 2\ 1\ 2\ 2\ 2\ 1\ 0.$$

The distinct nonzero letters in code(π) are $\{1, 2, 6\}$. Thus, dmc(π) = 3.

Dumont showed bijectively that the statistic dmc is Eulerian. While few researchers have found an application for Dumont's statistic since [6], Foata [8] proved the following equidistribution result involving the statistics INV (inversions) and MAJ (major index). These two statistics belong to the class of *Mahonian* statistics. (See [8] for further information.)

Theorem 1.1. *The Eulerian-Mahonian statistic pairs* (des, INV) *and* (dmc, MAJ) *are equally distributed on S_n, i.e.*

$$\#\{\pi \in S_n \mid \text{des}(\pi) = k; \text{INV}(\pi) = p\} = \#\{\pi \in S_n \mid \text{dmc}(\pi) = k; \text{MAJ}(\pi) = p\}.$$

Note that each of the statistics des, exc, and dmc is defined in terms of set cardinalities. We denote the *descent set* and *excedance set* of a permutation π by $D(\pi)$ and $E(\pi)$, respectively. We define the *letter set* of an arbitrary word to be the set of its nonzero letters, and denote the letter set of code(π) by $LC(\pi)$. Thus,

$$\text{des}(\pi) = |D(\pi)|,$$
$$\text{exc}(\pi) = |E(\pi)|,$$
$$\text{dmc}(\pi) = |LC(\pi)|.$$

It is easy to see that for every subset T of $[n-1] = \{1, \dots, n-1\}$, there are permutations π, σ, and ρ in S_n satisfying

$$T = D(\pi) = E(\sigma) = LC(\rho).$$

In fact, Dumont's original bijection [6] shows that for each such subset T we have

$$\#\{\pi \in S_n \,|\, E(\pi) = T\} = \#\{\pi \in S_n \,|\, LC(\pi) = T\}.$$

However, the analogous statement involving $D(\pi)$ is not true.

Generalizing permutations on n letters are *words* $w = w_1 \cdots w_m$ on n letters, where $m \geq n$. We will assume that each letter in $[n]$ appears at least once in w. Generalizing the symmetric group S_n, we define the *rearrangement class* of w by

$$R(w) = \{w_{\sigma^{-1}(1)} \cdots w_{\sigma^{-1}(m)} \,|\, \sigma \in S_m\}.$$

Each element of $R(w)$ is called a *rearrangement* of w.

Many definitions pertaining to S_n generalize immediately to the rearrangement class of any word. In particular, the definitions of descent, descent set, code, letter set of a code, and Dumont's statistic remain the same for words as for permutations. Generalization of excedances requires only a bit of effort.

For any word w, denote by $\bar{w} = \bar{w}_1 \cdots \bar{w}_m$ the unique *nondecreasing rearrangement* of w. We define position i to be an *excedance* in w if $w_i > \bar{w}_i$. Thus,

$$\text{exc}(w) = \#\{i \,|\, w_i > \bar{w}_i\}.$$

If position i is an excedance in word w, we will refer to the letter w_i as the *value* of excedance i. One can see word excedances most easily by associating to the word w the *biword*

$$\binom{\bar{w}}{w} = \binom{\bar{w}_1 \cdots \bar{w}_m}{w_1 \cdots w_m}$$

Example 1.2. Let $w = 312312311$. Then,

$$\binom{\bar{w}}{w} = \binom{1\ 1\ 1\ 1\ 2\ 2\ 3\ 3\ 3}{3\ 1\ 2\ 3\ 1\ 2\ 3\ 1\ 1}.$$

Thus, $E(w) = \{1, 3, 4\}$ and $\text{exc}(w) = 3$. The corresponding excedance values are 3, 2, and 3.

We will use biwords not only to expose excedances, but to define and justify maps in Sections 3 and 4. In particular, if $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_m$ are words and y is the biword

$$y = \binom{u}{v},$$

then we will define *biletters* $y_1, \ldots, y_m$ by

$$y_i = \binom{u_i}{v_i},$$

and will define the rearrangement class of y by

$$R(y) = \{y_{\sigma^{-1}(1)} \cdots y_{\sigma^{-1}(m)} \,|\, \sigma \in S_m\}.$$

A well known result concerning word statistics is that the statistics des and exc are equally distributed on the rearrangement class of any word w,

$$\#\{y \in R(w) \,|\, \mathrm{exc}(y) = k\} = \#\{y \in R(w) \,|\, \mathrm{des}(y) = k\}.$$

Analogously to the case of permutation statistics, a word statistic *stat* is called *Eulerian* if it satisfies

$$\#\{y \in R(w) \,|\, \mathrm{stat}(y) = k\} = \#\{y \in R(w) \,|\, \mathrm{des}(y) = k\}$$

for any word w and any nonnegative integer k.

In Section 2, we state and prove our main result: that dmc is Eulerian as a word statistic. Our bijection is different than that of Dumont [6], which doesn't generalize obviously to the case of arbitrary words. Applying the main theorem to a problem involving f-vectors and h-vectors of partially ordered sets, we state a second theorem in Section 3. This result strengthens a special case of a result of Stanley [9] concerning the flag h-vectors of balanced Cohen-Macaulay complexes. We prove the second theorem in Sections 4 and 5, and finish with some related open questions in Section 6.

2 Main theorem

As implied in Section 1, we define Dumont's statistic on an arbitrary word w to be the number of distinct nonzero letters in code(w).

$$\mathrm{dmc}(w) = |LC(w)|.$$

This generalized statistic is Eulerian.

Theorem 2.1. *If $R(w)$ is the rearrangement class of an arbitrary word w and k is any nonnegative integer, then*

$$\#\{v \in R(w) \,|\, \mathrm{dmc}(v) = k\} = \#\{v \in R(w) \,|\, \mathrm{exc}(v) = k\}.$$

Our bijective proof of the theorem depends upon an encoding of a word which we call the *excedance table.*

Definition 2.1. Let $v = v_1 \cdots v_m$ be an arbitrary word and let let $c = c_1 \cdots c_m$ be its code. Define the excedance table of v to be the unique word $\mathrm{etab}(v) = e_1 \cdots e_m$ satisfying

1. If i is an excedance in v, then $e_i = i$.
2. If $c_i = 0$, then $e_i = 0$.
3. Otherwise, e_i is the c_ith excedance of v having value at least v_i.

Note that $\mathrm{etab}(v)$ is well defined for any word v. In particular, if i is not an excedance in v and if $c_i > 0$, then there are at least c_i excedances in v having value at least v_i. To see this, define

$$k = \#\{j \in [m] \,|\, v_j < v_i\}.$$

Since c_i of the letters $\bar{v}_1, \ldots, \bar{v}_k$ appear to the right of position i in v, then at least c_i of the letters $\bar{v}_{k+1}, \ldots, \bar{v}_m$ must appear in the first k positions of v. The positions of these letters are necessarily excedances in v. An important property of the excedance table is that the letter set of $\mathrm{etab}(v)$ is precisely the excedance set of v.

Example 2.2. Let $v = 514514532$, and define $c = \mathrm{code}(v)$. Using v, $\bar{v}$, and c, we calculate $e = \mathrm{etab}(v)$,

$$\begin{aligned}
\bar{v} &= 1\ 1\ 2\ 3\ 4\ 4\ 5\ 5\ 5,\\
v &= 5\ 1\ 4\ 5\ 1\ 4\ 5\ 3\ 2,\\
c &= 6\ 0\ 3\ 4\ 0\ 2\ 2\ 1\ 0,\\
e &= 1\ 0\ 3\ 4\ 0\ 3\ 4\ 1\ 0.
\end{aligned}$$

Calculation of $e_1, \ldots, e_5$ and e_9 is straightforward since the positions $i = 1, \ldots, 5$ and 9 are excedances in v or satisfy $c_i = 0$. We calculate e_6, e_7, and e_8 as follows. Since $c_6 = 2$, and the second excedance in v with value at least $v_6 = 4$ is 3, we set $e_6 = 3$. Since $c_7 = 2$, and the second excedance in v with value at least $v_7 = 5$ is 4, we set $e_7 = 4$. Since $c_8 = 1$, and the first excedance in v with value at least $v_8 = 3$ is 1, we set $e_8 = 1$.

We prove Theorem 2.1 with a bijection $\theta : R(w) \rightarrow R(w)$ which satisfies

$$E(v) = LC(\theta(v)), \tag{2.1}$$

and therefore

$$\mathrm{exc}(v) = \mathrm{dmc}(\theta(v)). \tag{2.2}$$

Definition 2.3. Let $w = w_1 \cdots w_m$ be any word. Define the map $\theta : R(w) \rightarrow R(w)$ by applying the following procedure to an arbitrary element v of $R(w)$.

1. Define the biword $z = \binom{v}{\mathrm{etab}(v)}$.
2. Let y be the unique rearrangement of z satisfying $y = \binom{u}{\mathrm{code}(u)}$.
3. Set $\theta(v) = u$.

Construction of y is quite straightforward. Let $e = e_1 \cdots e_m = \mathrm{etab}(v)$, and linearly order the biletters $z_1, \ldots, z_m$ by setting $z_i < z_j$ if

$$\begin{aligned} &v_i < v_j, \text{ or} \\ &v_i = v_j \text{ and } e_i > e_j. \end{aligned}$$

Break ties arbitrarily. Considering the biletters according to this order, insert each biletter z_i into y to the left of e_i previously inserted biletters.

Example 2.4. Let v and e be as in Example 2.2. To compute $\theta(v)$, we define

$$z = \binom{v}{e} = \begin{pmatrix} 5\,1\,4\,5\,1\,4\,5\,3\,2 \\ 1\,0\,3\,4\,0\,3\,4\,1\,0 \end{pmatrix}.$$

We consider the biletters of z in the order

$$\binom{1}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{1}, \binom{4}{3}, \binom{4}{3}, \binom{5}{4}, \binom{5}{4}, \binom{5}{1},$$

and insert them individually into y:

$$\binom{1}{0}, \begin{pmatrix} 1\,1 \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 1\,1\,2 \\ 0\,0\,0 \end{pmatrix}, \begin{pmatrix} 1\,1\,3\,2 \\ 0\,0\,1\,0 \end{pmatrix}, \begin{pmatrix} 1\,4\,1\,3\,2 \\ 0\,3\,0\,1\,0 \end{pmatrix}, \ldots$$

Finally we obtain

$$y = \binom{u}{\mathrm{code}(u)} = \begin{pmatrix} 1\,4\,5\,5\,4\,1\,3\,5\,2 \\ 0\,3\,4\,4\,3\,0\,1\,1\,0 \end{pmatrix}$$

and set $\theta(v) = 145541352$.

It is easy to see that any biword z has at most one rearrangement y satisfying Definition 2.3 (2). Such a rearrangement exists if and only if we have

$$e_i \leq \#\{j \in [m] \mid v_j < v_i\}, \text{ for } i = 1, \ldots, m, \tag{2.3}$$

or equivalently, if and only if

$$\bar{v}_{e_i} < v_i, \text{ for } i = 1, \ldots, m, \tag{2.4}$$

where we define $\bar{v}_0 = 0$ for convenience.

Observation 2.2. *Let $v = v_1 \cdots v_m$ be any word and let $e = \mathrm{etab}(v)$. Then we have*

$$e_i \leq \#\{j \in [m] \mid v_j < v_i\}, \text{ for } i = 1, \ldots, m.$$

Proof. If i is an excedance in v, then $e_i = i$ and $\bar{v}_1 \leq \cdots \leq \bar{v}_i < v_i$. If $c_i = 0$, then $e_i = 0$. Otherwise, define

$$k = \#\{j \in [m] \mid v_j < v_i\}.$$

By the discussion following Definition 2.1, at least c_i of the positions $1, \ldots, k$ are excedances in v with values at least v_i. The letter e_i, being one of these excedances, is therefore at most k. □

Thus the map θ is well defined and satisfies (2.1) and (2.2). We invert θ by applying the procedure in the following proposition.

Proposition 2.3. *Let $y = \binom{u}{c}$ be a biword satisfying $c = \mathrm{code}(u)$. The following procedure produces a rearrangement $z = \binom{v}{e}$ of y satisfying $e = \mathrm{etab}(v)$.*

1. *For each letter ℓ in $L(c)$, find the greatest index i satisfying $c_i = \ell$, and define $z_\ell = y_i$. Let S be the set of such greatest indices, and define $T = [m] \smallsetminus S$.*

2. *For each index $i \in T$, define*

$$d_i = \begin{cases} \#\{j \in S \mid c_j \leq c_i; u_j \geq u_i\}, & \text{if } c_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

3. *Let $(y_{\sigma^{-1}(i)})_{i \in T}$ be the unique rearrangement of $(y_i)_{i \in T}$ satisfying*

$$(d_{\sigma^{-1}(i)})_{i \in T} = \mathrm{code}((u_{\sigma^{-1}(i)})_{i \in T}).$$

4. *Insert the biletters $(y_{\sigma^{-1}(i)})_{i \in T}$ in order into the remaining positions of z.*

Proof. The procedure above is well defined. In particular, we may perform step 3 because the biword

$$\binom{u_i}{d_i}_{i \in T}$$

satisfies

$$d_i \leq \#\{j \in T \,|\, u_j < u_i\}, \text{ for each } i \in T,$$

as required by (2.3). To see that this is the case, let i be an index in T with $c_i > 0$. In step 1 we have placed d_i biletters y_j with $u_j > u_i$ into positions $1, \ldots, c_i$ of z. Thus, at least d_i biletters y_j with $u_j \leq \bar{u}_{c_i}$ have *not* been placed into these positions. The index j of any such biletter belongs to S only if $c_j > c_i$. However, since $\bar{u}_{c_j} < u_j \leq \bar{u}_{c_i} < u_i$, we have $c_j < c_i$. Thus, j belongs to T.

We begin to verify that $e = \mathrm{etab}(v)$ by calculating the excedance set of v. We claim that $E(v) = L(c)$.

The positions $L(c) = \{c_j \,|\, j \in S\}$ are excedances in v, because for each index j in S, we have $v_{c_j} = u_j > \bar{u}_{c_j} = \bar{v}_{c_j}$. These positions are in fact the only excedances in v. For each index j in T, denote by $\phi(j)$ the position of z into which we have placed y_j. Supposing that some indices $\{\phi(j) \,|\, j \in T\}$ are excedances in v, choose $i \in T$ so that $\phi(i)$ is the leftmost of these excedances and define

$$k = \#\{j \in [m] \,|\, u_j < u_i\}.$$

Then, we have

$$k > \#\{j \in S \,|\, c_j \leq k\} + \#\{j \in T \,|\, \sigma(j) < \sigma(i)\}. \tag{2.5}$$

Since $c_i \leq k$ by (2.3), we have

$$\#\{j \in S \,|\, c_j \leq k\} = \#\{j \in S \,|\, c_j \leq c_i\} + \#\{j \in S \,|\, c_i < c_j \leq k\},$$

and by the definition of σ, we have

$$\begin{aligned}\#\{j \in T \,|\, \sigma(j) < \sigma(i)\} &= \#\{j \in T \,|\, u_j < u_i\} - d_i \\ &= \#\{j \in T \,|\, u_j < u_i\} - \#\{j \in S \,|\, c_j \leq c_i; u_j \geq u_i\}.\end{aligned}$$

Using these identities to simplify (2.5), we obtain

$$\#\{j \in S \,|\, u_j < u_i; c_j > c_i\} > \#\{j \in S \,|\, c_i < c_j \leq k\}. \tag{2.6}$$

If j belongs to the set on the left hand side of (2.6) and satisfies $c_j > k$, then we have

$$u_j > \bar{u}_{c_j} \geq \bar{u}_k = u_i - 1,$$

a contradiction. On the other hand, if each index j in this set satisfies $c_j \leq k$, then we have the inclusion

$$\{j \in S \,|\, u_j < u_i; c_j > c_i\} \subset \{j \in S \,|\, c_i < c_j \leq k\},$$

which contradicts (2.6). We conlcude that the set $\{\phi(j) \,|\, j \in T\}$ is precisely the set of non-excedances in v, and that we have

$$E(v) = L(c) = \{c_j \,|\, j \in S\}.$$

Finally, we show that e has the defining properites of $\mathrm{etab}(v)$. For each index j in S, we have defined $e_{c_j} = c_j$ so that e satisfies condition (1) of Definition 2.1. Let c' be the code of v. We claim that for each index $i \in T$, we have

$$e_{\phi(i)} = c_i = \begin{cases} \text{the } c'_{\phi(i)}\text{th excedance in } v \text{ having value at least } u_i, & \text{if } c'_{\phi(i)} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By our definition of the sequence $(d_i)_{i \in T}$, it suffices to show that $c'_{\phi(i)} = d_i$ for each index i. The subword $v_{\phi(i)+1} \cdots v_m$ of v includes d_i letters $v_{\phi(j)}$ with $j \in T$ and $v_{\phi(j)} < v_{\phi(i)}$. On the other hand, any excedance in v to the right of $\phi(i)$ has value *greater* than $v_{\phi(i)}$. We conclude that $c'_{\phi(i)} = d_i$. □

The above procedure inverts θ because the biword z it produces is the *unique* rearrangement of y having the desired properties.

Proposition 2.4. *Let $v = v_1 \cdots v_m$ be an arbitrary word, and define*

$$z = \binom{v}{e} = \binom{v}{\mathrm{etab}(v)}.$$

If there is any rearrangement z' of z satisfying

$$z' = \binom{v'}{e'} = \binom{v'}{\mathrm{etab}(v')},$$

then $z' = z$.

Proof. Let L be the letter set of e. By Definition 2.1, we must have $E(v) = E(v') = L$. Let i be an excedance of v and v'. By condition (1) of Definition 2.1 we must have $e_i = e'_i = i$, and by condition (3) the upper letters v_i and v'_i must be as large as possible. Thus, $(z_i)_{i \in L} = (z'_i)_{i \in L}$.

Let $T = [m] \smallsetminus L$ be the set of non-excedance positions of v and v', and consider the corresponding subsequences of biletters $(z_i)_{i \in T}$ and $(z'_i)_{i \in T}$. By condition (3) of Definition 2.1, the codes of $(v_i)_{i \in T}$ and $(v'_i)_{i \in T}$ are determined by the excedances and excedance values in v and v'. Thus, the two codes must be identical. Applying the argument following Example 2.4, we conclude that $(z_i)_{i \in T} = (z'_i)_{i \in T}$. □

3 An application of Dumont's statistic

As an application of Dumont's (generalized) statistic, we will strengthen a special case of a result of Stanley [9, Cor. 4.5] concerning f-vectors and h-vectors of simplicial complexes.

Given a $(d-1)$ dimensional simplicial complex Σ, we define its f-vector to be

$$f_\Sigma = (f_{-1}, f_0, f_1, \dots, f_{d-1}),$$

where f_i counts the number of i-dimensional faces of Σ. By convention, $f_{-1} = 1$. Similarly, we may define the f-vector of a poset P by identifying P with its *order complex* $\Delta(P)$. (See [10, p. 120].) That is, we define

$$f_P = f_{\Delta(P)} = (f_{-1}, f_0, f_1, \dots, f_{d-1}),$$

where f_i counts the number of $(i+1)$-element chains of P. Again, $f_{-1} = 1$ by convention.

In abundant research papers, authors have considered the f-vectors of various classes of complexes and posets, and have conjectured or obtained significant information about the coefficients. (See [1], [2], [11, Ch. 2,3].) Such information includes linear relationships between coefficients and properties such as symmetry, log concavity and unimodality.

Related to the f-vector f_Σ is the h-vector $h_\Sigma = (h_0, h_1, \dots, h_d)$, which we define by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

From this definition, it is clear that knowing the h-vector of a complex is equivalent to knowing the f-vector. For some conditions on a simplicial complex, one can show that its h-vector is the f-vector of another complex. Specifically, we have the following result due to Stanley [9, Cor. 4.5].

Theorem 3.1. *If Σ is a balanced Cohen-Macaulay complex, then its h-vector is the f-vector of some simplicial complex Γ.*

We define a simplicial complex to be *Cohen-Macaulay* if it satisfies a certain topological condition ([11, p. 61]), and *balanced* if we can color the vertices with d colors such that no face contains two vertices of the same color ([11, p. 95]). The class of balanced Cohen-Macaulay complexes is quite important because it includes the order complexes of all distributive lattices. The distributive lattices, in turn, contain information about all posets. (See [10, Ch. 3].)

By placing an additional restriction on the complex Σ, one arrives at a special case of the theorem which has an elegant bijective proof. Let us require that Σ be the order complex of a distributive lattice $J(P)$. In this case, $h_\Sigma = h_{J(P)}$ counts the number of linear extensions of P by descents. (See [4].) That is, h_k is the number of linear extensions of P with k descents. Therefore, Theorem 3.1 asserts that for any poset P, there is a bijective correspondence between linear extensions of P with k descents and $(k-1)$-faces of some simplicial complex Γ.

$$\{\pi \mid \pi \text{ a linear extension of } P;\, \mathrm{des}(\pi) = k\} \overset{1-1}{\longleftrightarrow} \{\sigma \mid \sigma \text{ a } (k-1)\text{-face of } \Gamma\}.$$

Using [3, Remark 6.6] and [7, Cor. 2.2], one can construct a family $\{\Xi_n\}_{n>0}$ of simplicial complexes such that for any poset P on n elements, the complex Γ corresponding to $\Sigma = \Delta(J(P))$ is a subcomplex of Ξ_n.

On the other hand, any additional restriction placed on the complex Σ in Theorem 3.1 should allow us to prove more than a special case of the theorem. It should allow us to strengthen the special case by asserting specific properties of the complex Γ in the conclusion of the theorem. In particular, let us require that Σ be the order complex of a distributive lattice $J(P)$ which is a product of chains. (See [10, Ch. 3] for definitions.) We will prove the following result.

Theorem 3.2. *Let the distributive lattice $J(P)$ be a product of chains. Then there is a poset Q such that the h-vector of $J(P)$ is the f-vector of Q.*

Let us reconsider this theorem in terms of rearrangements of words. If $J(P)$ is a product of chains having cardinalities $(p_1+1), \ldots, (p_n+1)$, then P is the disjoint *sum* of chains $(\mathbf{p_1} + \cdots + \mathbf{p_n})$. It is not difficult to see that linear extensions of P

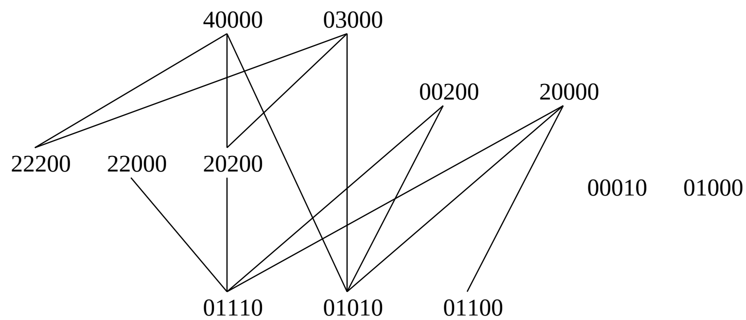

Figure 3.1: A poset with a k-element chain for each k-letter code in $C(11223)$.

are in bijective correspondence with rearrangements of the word $w = 1^{p_1} \cdots n^{p_n}$. Combining this observation with Theorem 2.1, we restate Theorem 3.2 in terms of Dumont's statistic.

Proposition 3.3. *Let w be any word and define the vector $h = (h_0, \ldots, h_d)$ by*

$$h_i = \#\{u \in R(w) \,|\, \mathrm{dmc}(u) = i\},$$

where d is the maximum cardinality of $LC(u)$ over all rearrangements u of w. Then, there is a poset Q whose f-vector is h.

To prove the proposition, and therefore Theorem 3.2, we will work directly with codes of rearrangements of a word. Let us denote $C(w)$ be the set of codes of all rearrangements of w. Proposition 3.3 asserts that for any word w, there is a bijection between k-letter elements of $C(w)$ and k-element chains in some poset Q,

$$\{c \in C(w) \,|\, c \text{ a } k\text{-letter code}\} \stackrel{1-1}{\longleftrightarrow} \{(v_1 <_Q \cdots <_Q v_k) \in \Delta(Q)\}. \tag{3.1}$$

We will construct such a poset $Q = Q(w)$ as follows.

Definition 3.1. Given an arbitrary word w, let Q be the subset of one-letter codes in $C(w)$. For each pair (c, c') of codes in Q whose letters are (ℓ, ℓ'), respectively, define $c <_Q c'$ if

1. $\ell < \ell'$.
2. The multiplicity of ℓ in c is strictly *greater* than that of ℓ' in c'.
3. For each position i such that $c'_i = \ell'$, we have $c_{i+\ell'-\ell} = \ell$.

Example 3.2. Figure 3.1 shows the poset Q corresponding to the word $w = 11223$. The f-vector f_Q counts words in $R(w)$ by Dumont's statistic. Equivalently, it counts linear extensions of the poset $P = \mathbf{2} + \mathbf{2} + \mathbf{1}$ by descent, and is equal to the h-vector of $J(P)$,

$$f_Q = h_{J(P)} = (1, 12, 15, 2).$$

In Sections 4 and 5 we will demonstrate that for any word w, the procedure in Definition 3.1 gives a poset Q satisfying the bijections of (3.1). We will give an explicit bijection $\Psi : C(w) \to \Delta(Q)$, taking k-letter codes in $C(w)$ to k-element chains in Q.

4 The chain map Ψ

Fix a nondecreasing word $w = w_1 \cdots w_m$ on n letters, and define the poset Q as in Definition 3.1. We will define a *chain map* $\Psi : C(w) \to \Delta(Q)$ which will identify a code c with a chain

$$\Psi(c) = v_1 <_Q \cdots <_Q v_k,$$

of elements in Q. If c is a code on the k letters $\ell_1 < \cdots < \ell_k$, then each poset element v_i will be a code whose unique nonzero letter is ℓ_i. Specifically, we will determine v_i by applying a *vertex map* $\psi_{\ell_i} : C(w) \to Q$ to c.

$$v_i = \psi_{\ell_i}(c).$$

After proving that $\psi_{\ell_i}(c) <_Q \psi_{\ell_j}(c)$ whenever $\ell_i < \ell_j$, we will define the chain map to be a product of vertex maps,

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c).$$

We begin by observing that several simple operations on codes in $C(w)$ yield other codes in $C(w)$.

Observation 4.1. *Let u be a rearrangement of w and let $c = \mathrm{code}(u)$.*

1. *If $c_i > c_{i+1}$, then the word*

$$c' = c_1 \cdots c_{i-1} \cdot c_{i+1} \cdot (c_i - 1) \cdot c_{i+2} \cdots c_m$$

belongs to $C(w)$.

2. *If for some $r > i$, c_i is strictly less than $c_{i+1}, \ldots, c_r$ and $c_i > c_{r+1}$, then the word*

$$c'' = c_1 \cdots c_{i-1} \cdot c_{r+1} \cdot c_{i+1} \cdots c_r \cdot (c_i - 1) \cdot c_{r+2} \cdots c_m$$

belongs to $C(w)$.

3. *If $c_i < c_{i+1}$, or if $c_i = c_{i+1}$ and $u_i < u_{i+1}$, then the word*

$$c''' = c_i \cdots c_{i-1} \cdot (c_{i+1} + 1) \cdot c_i \cdot c_{i+2} \cdots c_m$$

belongs to $C(w)$.

Proof. Let u' be the word obtained from u by switching the letters in positions i and $i+1$, and let u'' be the word obtained by switching the letters in positions i and $r+1$,

$$u' = (u_1 \cdots u_{i-1} \cdot u_{i+1} \cdot u_i \cdot u_{i+2} \cdots u_m),$$
$$u'' = (u_1 \cdots u_{i-1} \cdot u_{r+1} \cdot u_{i+1} \cdots u_r \cdot u_i \cdot u_{r+2} \cdots u_m).$$

(1) We have $u_i > u_{i+1}$ and $c' = \text{code}(u')$.

(2) We have $u_{r+1} < u_i < u_{i+1}, \cdots, u_r$ and $c'' = \text{code}(u'')$.

(3) We have $u_i < u_{i+1}$ and $c''' = \text{code}(u')$.

□

Using this observation we will define two families of maps from $C(w)$ to itself, $\lambda_1, \ldots, \lambda_{m-1}$ and $\mu_1, \ldots, \mu_{m-1}$. Then, composing maps from these two families, we will define the family of vertex maps $\psi_1, \ldots, \psi_{m-1}$.

The map $\lambda_{\ell_i} : C(w) \to C(w)$ removes from a code c all letters ℓ_j which are greater than ℓ_i. It essentially changes each such letter ℓ_j to ℓ_i and moves it $\ell_j - \ell_i$ places to the right in c. If we identify c with the k-element chain $v_1 <_Q \cdots <_Q v_k$, then we will identify $\lambda_{\ell_i}(c)$ with the i-element subchain $v_1 <_Q \cdots <_Q v_i$.

Definition 4.1. Let ℓ be a nonzero letter. Define the map $\lambda_\ell : C(w) \to C(w)$ by performing the following procedure on a code c.

For $i = m, m-1, \ldots, 1$, if $c_i > \ell$, then

1. Set $\delta = c_i - \ell$.

2. Redefine $c = c_1 \cdots c_{i-1} \cdot c_i \cdots c_{i+\delta} \cdot \ell \cdot c_{i+\delta+1} \cdots c_m$.

Analogous to λ_{ℓ_i}, the map $\mu_{\ell_i} : C(w) \to C(w)$ removes all letters which are *smaller* than ℓ_i. It does so by changing each such smaller letter to 0. If we identify c with the k-element chain $v_1 <_Q \cdots <_Q v_k$, then we will identify $\mu_{\ell_i}(c)$ with the $(k-i+1)$-element subchain $v_i <_Q \cdots <_Q v_k$.

Definition 4.2. Let ℓ be a nonzero letter. Define the map $\mu_\ell : C(w) \to C(w)$ by $\mu_\ell(c) = a_1 \cdots a_m$, where

$$a_i = \begin{cases} 0, & \text{if } c_i < \ell, \\ c_i, & \text{otherwise.} \end{cases}$$

The maps $\lambda_1, \ldots, \lambda_{m-1}$, and $\mu_1, \ldots, \mu_{m-1}$ are well defined, for their definitions are merely repeated applications of Observation 4.1 (1) and (2). Note that the composition $\mu_\ell \lambda_\ell$ produces a code on the single letter ℓ. This code is an element of Q, and a *vertex* of $\Delta(Q)$.

Definition 4.3. Let ℓ be a nonzero letter. Define the *vertex map* $\psi_\ell : C(w) \to Q$ by

$$\psi_\ell = \mu_\ell \lambda_\ell.$$

It is easy to see that $\lambda_\ell^2 = \lambda_\ell$, and therefore that $\psi_\ell \lambda_\ell = \psi_\ell$. These and the following relations will be essential in establishing a bijection between $C(w)$ and $\Delta(Q)$.

Proposition 4.2. *Let ℓ and ℓ' be letters, $1 \le \ell < \ell' \le n$. The maps $\lambda_\ell, \lambda_{\ell'}, \psi_\ell$, and $\psi_{\ell'}$ satisfy the relations*

1. $\lambda_{\ell'} \lambda_\ell = \lambda_\ell \lambda_{\ell'} = \lambda_\ell$.
2. $\psi_\ell \lambda_{\ell'} = \psi_\ell$.
3. $\psi_\ell(c) <_Q \psi_{\ell'}(c)$, *if c contains both letters.*

Proof. (1) Let $c = \text{code}(u)$ be an element of $C(w)$. By the comments following Definition 4.2, we may interpret $\lambda_\ell(c)$ as follows. Define $b = b_1 \cdots b_m$ by

$$b_i = \begin{cases} \ell, & \text{if } c_i > \ell, \\ c_i, & \text{otherwise,} \end{cases}$$

and rearrange the biword $\binom{u}{b}$ as $\binom{u'}{b'}$ so that $b' = \text{code}(u')$. Then, $b' = \lambda_\ell(c)$.

It is not hard to see that there is a unique such rearrangement. Using this interpretation, it is easy to see that $\lambda_{\ell'}\lambda_\ell$, $\lambda_\ell\lambda_{\ell'}$, and λ_ℓ describe the same procedure.

(2) Using (1), we have $\psi_\ell\lambda_{\ell'} = \mu_\ell\lambda_\ell\lambda_{\ell'} = \mu_\ell\lambda_\ell = \psi_\ell$.

(3) We may assume that ℓ' is the greatest letter in c. (Otherwise, we define $d = \lambda_{\ell'}(c)$ and note that $\psi_\ell(c) = \psi_\ell(d)$ and $\psi_{\ell'}(c) = \psi_{\ell'}(d)$.) Let $e = \psi_\ell(c)$ and $e' = \psi_{\ell'}(c)$. Clearly, the multiplicity of ℓ in e is strictly greater than that of ℓ' in e', for

$$\#\{i \,|\, e_i = \ell\} = \#\{i \,|\, c_i \geq \ell\} > \#\{i \,|\, c_i \geq \ell'\} = \#\{i \,|\, e'_i = \ell'\}.$$

Next, we show that for any position i of e' satisfying $e'_i = \ell$, we must have $e_{i+\ell'-\ell} = \ell$. Since by assumption, ℓ' is the greatest letter in c, we have $e'_i = \ell'$ if and only if $c_i = \ell'$. To find e, we first calculate $\lambda_\ell(c)$ by the procedure of Definition 4.1. At each iteration i such that $c_i = \ell'$, we place the letter ℓ into position $i + \ell' - \ell$ of $\lambda_\ell(c)$. This position will not be altered by iterations $i-1, \ldots, 1$, since all letters of c are no greater than ℓ'. Finally, since $e = \mu_\ell\lambda_\ell(c)$, and μ_ℓ changes only those letters less than ℓ, we see that $e_{i+\ell'-\ell} = \ell$ for every position i such that $e_i = \ell'$. □

Now we may define the map Ψ.

Definition 4.4. Define the *chain* map $\Psi : C(w) \to \Delta(Q)$ by

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c),$$

where $\ell_1 < \cdots < \ell_k$ are the distinct nonzero letters in c.

5 Inverting Ψ

We will define a map $\Phi : \Delta(Q) \to C(w)$ which takes a k-element chain in Q to a k-letter code in $C(w)$. By demonstrating that Φ inverts Ψ, we will complete the proof of Proposition 3.3.

We begin by defining an operation $\vee : C(w) \times Q \to C(w)$ which joins a new letter to a code.

Definition 5.1. Let $d \in Q$ be a code whose unique nonzero letter is ℓ', and let $c \in C(w)$ be a code whose greatest letter is ℓ. Assume that $\psi_\ell(c) <_Q d$. Define the code $e = c \vee d$ by the following procedure.

1. For each i such that $d_i = \ell'$, set $e_i = \ell'$ and cross out the ℓ in position $i + \delta$ of c.

2. Fill the remaining positions of e with the remaining components of c, in order.

Note that $L(e) = L(c) \cup \{\ell\}$. Therefore, we may map a chain of k one-letter codes to a single k-letter code by iterating the join operation.

Definition 5.2. Let $v_1 <_Q \cdots <_Q v_k$ be a chain of one-letter codes on the letters $\ell_1 < \cdots < \ell_k$, respectively. Define the map $\Phi : \Delta(Q) \to C(w)$ by

$$\Phi(v_1 <_Q \cdots <_Q v_k) = (\cdots((v_1 \vee v_2) \vee v_3) \cdots) \vee v_k.$$

The following proposition shows that the join operation is well defined. It follows that Φ is well defined also.

Proposition 5.1. *If c and d are codes in $C(w)$ satisfying the hypotheses of Definition 5.1, then $c \vee d$ also belongs to $C(w)$.*

Proof. Let u and y be words in $R(w)$ whose codes $c = \text{code}(u)$ and $d = \text{code}(y)$ satisfy the conditions of Definition 5.1. Consider the leftmost position i in c such that $c_i = \ell$ and $d_{i-\delta} = \ell'$. By assumption, $c_{i-1} \leq c_i$. If $c_{i-1} < c_i$, or if $c_{i-1} = c_i$ and $u_{i-1} < u_i$, then we may apply Observation 4.1 (3) $\ell' - \ell$ times to obtain the word

$$c_1 \cdots c_{i-\delta-1} \cdot \ell' \cdot c_{i-\delta+1} \cdots \hat{c}_i \cdots c_m,$$

which belongs to $C(w)$. (Here, $\hat{c}_i$ means that the letter c_i is omitted.) Repeating this process for each such position i, we redefine the join operation. Therefore it suffices to show that for every position i satisfying $d_{i-\delta-1} = 0$, $d_{i-\delta} = \ell'$, and $c_{i-1} = c_i = \ell$, we have $u_{i-1} < u_i$.

Let i be such a position and suppose that $u_{i-1} = u_i$. Since $d_{i-\delta} = \ell'$ and $d_{i-\delta-1} = 0$, there are exactly $i - \delta - 1 + \ell' = i + \ell - 1$ letters in y which are strictly less than y_i. In particular, we have

$$w_{i+\ell-1} < w_{i+\ell}. \tag{5.1}$$

Let k be the number of positions preceeding i such that $u_{i-k} = u_{i-k+1} = \cdots = u_i$ and $c_{i-k} = c_{i-k+1} = \cdots = \ell$. Then there are exactly $i - k - 1 + \ell$ letters in u which are strictly less than u_i $(= u_{i-1} = \cdots = u_{i-k})$. In particular, we have

$$w_{i-k-1+\ell} < w_{i-k+\ell} = w_{i-k+1+\ell} = \cdots = w_{i+\ell},$$

which contradicts (5.1). We conclude that $u_{i-1} < u_i$, and therefore that $c \vee d$ belongs to $C(w)$. □

To begin demonstrating that Φ inverts Ψ we note the following relations satisfied by ψ, λ and $\vee$.

Proposition 5.2. *The pair of maps (ψ, λ) inverts the operation $\vee$ in the following sense.*

1. *Let $c \in C(w)$ be a code with greatest letter ℓ, and let $d \in Q$ be a code with letter $\ell' > \ell$ and satisfying $\psi_\ell(c) <_Q d$. Then we have*

$$\psi_{\ell'}(c \vee d) = d,$$
$$\lambda_\ell(c \vee d) = c.$$

2. *Let $c \in C(w)$ be a code whose greatest two letters are $\ell < \ell'$. Then we have*

$$\lambda_\ell(c) \vee \psi_{\ell'}(c) = c.$$

Proof. (1) Let S be the set of positions of d containing the letter ℓ', and let $\delta = \ell' - \ell$.

Define the words $e = c \vee d$, $d' = \psi_{\ell'}(c \vee d)$, and $c' = \lambda_\ell(c \vee d)$. Calculating e, we have

$$(e_i)_{i \in S} = \ell' \cdots \ell',$$
$$(e_i)_{i \notin S} = (c_i)_{i-\delta \notin S}.$$

Since e contains no letters greater than ℓ', we have $d' = \psi_{\ell'}(e) = \mu_{\ell'}(e)$. Thus, $d' = d$:

$$(d'_i)_{i \in S} = \ell' \cdots \ell',$$
$$(d'_i)_{i \notin S} = 0 \cdots 0.$$

Calculating $c' = \lambda_\ell(e)$, we change each occurrence of ℓ' in e to ℓ, and move it δ positions to the right. Since $\psi_\ell(c) <_Q d$, we see that $c' = c$:

$$(c'_i)_{i-\delta \in S} = \ell \cdots \ell = (c_i)_{i-\delta \in S},$$
$$(c'_i)_{i-\delta \notin S} = (c_i)_{i-\delta \notin S}.$$

(2) Similar. □

Completing the proof of Proposition 3.3, the following proposition shows that Ψ is bijective.

Proposition 5.3. *Let $c \in C(w)$ be a code on the letters $\ell_1 < \cdots < \ell_k$, and let $v_1 <_Q \cdots <_Q v_k$ be a k-element chain in Q, where the letter of v_i is ℓ_i for each i. The maps Ψ and Φ satisfy*

1. $\Psi\Phi(v_1 <_Q \cdots <_Q v_k) = v_1 <_Q \cdots <_Q v_k$.

2. $\Phi\Psi(c) = c$.

Proof. (1) By Definition 5.2, we have

$$\Phi(v_1 <_Q \cdots <_Q v_k) = (\cdots((v_1 \vee v_2) \vee v_3) \cdots) \vee v_k.$$

Applying $\Psi = \psi_{\ell_1} \times \cdots \times \psi_{\ell_k}$ to this code, we calculate $\psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k)$, for $i = 1, \ldots, k$. By Proposition 5.2 (1), we have

$$\begin{aligned}\psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k) &= \psi_{\ell_i}\lambda_{\ell_i}\lambda_{\ell_{i+1}} \cdots \lambda_{\ell_k}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k)\\ &= \psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_i)\\ &= v_i,\end{aligned}$$

as desired.

(2) By Definition 4.4, we have

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c).$$

Applying Φ to this chain, we join vertices one at a time. Noting that $\psi_{\ell_1}(c) = \lambda_{\ell_1}(c)$, we use Proposition 5.2 (2) to calculate

$$\begin{aligned}\lambda_{\ell_i}(c) \vee \psi_{\ell_{i+1}}(c) &= \lambda_{\ell_i}\lambda_{\ell_{i+1}}(c) \vee \psi_{\ell_{i+1}}\lambda_{\ell_{i+1}}(c)\\ &= \lambda_{\ell_i}(\lambda_{\ell_{i+1}}(c)) \vee \psi_{\ell_{i+1}}(\lambda_{\ell_{i+1}}(c))\\ &= \lambda_{\ell_{i+1}}(c).\end{aligned}$$

Thus, after $k - 1$ join iterations, we recover c. □

6 Open questions

Since the class of balanced Cohen-Macaulay complexes contains so many widely studied classes of complexes, there are many possibilities to refine Theorem 3.1. In Theorem 3.2, we have required that Σ be an order complex of the form $\Delta(J(P))$, where P is a disjoint sum of chains. One could also ask if the theorem holds for more general classes of posets. (See [10], [11] for definitions in the questions that follow.) For instance, the following questions are open.

Question 6.1. If P is a forest, then is there another poset Q such the h-vector of $J(P)$ is the f-vector of Q?

Question 6.2. If P is a series-parallel poset, then is there another poset Q such the h-vector of $J(P)$ is the f-vector of Q?

We conjecture that the answers to both questions are affirmative. In fact, we conjecture that the answer remains affirmative for *any* choice of a poset P.

Conjecture 6.1. *Let $J(P)$ be any distributive lattice. Then there is another poset Q such that the h-vector of $J(P)$ is the f-vector of Q.*

This conjecture has been tested by computer for all distributive lattices $J(P)$ arising from posets P having up to seven elements. Other open questions place requirements on Γ instead of on Σ.

Question 6.3. For which balanced Cohen-Macaulay complexes Σ is h_Σ the f-vector of a *graded* poset (or $(\mathbf{3}+\mathbf{1})$-free poset, or flag complex)?

To begin to answer Questions 6.1 - 6.3, it would be interesting to utilize any Eulerian permutation statistic *stat* to define posets such as Q in Definition 3.1 which satisfy the following two conditions.

1. For each k, the k-element chains in Q bijectively correspond to the linear extensions π of P with $\mathrm{stat}(\pi) = k$.
2. For each poset P in some class $\mathcal{P}$, the statistics stat and des are equidistributed on the set of linear extensions of P, so that $h_{J(P)} = f_Q$.

One might also consider a variation of this method based upon objects other than permutations, such as Motzkin paths or either of the tree representations in [10, pp. 23-25].

A result similar to Theorem 2.1 (in the sense that word rearrangements correspond to linear extensions of certain posets) states that the statistics INV and MAJ are equally distributed on the linear extensions of posets known as postorder labelled forests [5]. Perhaps Theorem 2.1 could be extended similarly.

Question 6.4. For what conditions on a poset P are the statistics des and dmc equidistributed on the set of linear extensions of P?

One might apply another variation of the method above by defining a rule which maps each n-element poset P to a subset $\mathcal{K}(P)$ of S_n which is *not* a set of linear extensions of P. This subset should have the property that the elements π in $\mathcal{K}(P)$ satisfying $\mathrm{stat}(\pi) = k$ are in bijective correspondence with the linear extensions of P which have k descents.

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