CHAIN POLYNOMIALS AND PERMUTATION STATISTICS

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Outline

1. Characterization of integer sequences
2. f-vectors and chain polynomials
3. Distributive lattices
4. h-vectors
5. Permutation statistics
6. $(\mathbf{3}+\mathbf{1})$-free posets

Sequence Characterization

Let $a = (a_0, \ldots, a_d)$ be a sequence of nonnegative integers.

Question. Under what conditions does the polynomial

$$a(x) = a_0 + a_1 x + \cdots + a_d x^d$$

have only real zeros?

(Thm of Gantmacher.)

Question. Under what conditions does the sequence a count faces of a simplicial complex?

(Thm of Schutzenberger, Kruskal, Katonah.)

Properties of sequences

The sequence $a = (a_0, \ldots, a_d)$ may have the following properties.

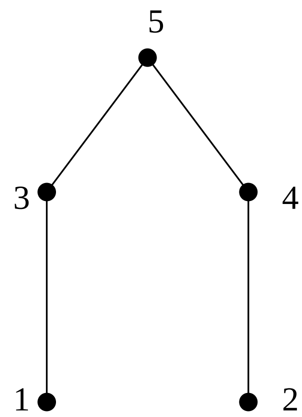
1. a is *unimodal*, i.e.,
$$a_0 \leq \cdots \leq a_j \geq \cdots \geq a_d,$$
for some j.

2. a is *log-concave*, i.e.,
$$a_i^2 \geq a_{i-1}a_{i+1},$$
for $i = 1, \ldots, d-1$.

3. The polynomial
$$a_0 + a_1 x + \cdots + a_d x^d$$
has only real zeros.

Two descriptions of a sequence

Theorem. *The sequence* $a = (a_0, \cdots, a_d)$ *is the* h*-vector of some Cohen-Macauley complex if and only if it counts faces in some multicomplex.*



f-vectors

The *f-vector* of a poset P,

$$f_P = (f_{-1}, f_0, \dots, f_{d-1}),$$

counts chains by cardinality. f_i is the number of $(i+1)$-element chains in P.

For the poset above, we have

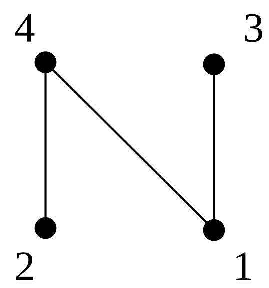
$$f_P = (1, 5, 6, 2).$$

We define the *chain polynomial* of P by

$$f_P(x) = f_{-1} + f_0 x + \cdots + f_{d-1} x^d$$

Question. Which sequences $(a_0, \ldots, a_d)$ are f-vectors of posets?
(Open.)

Question. For which posets P does the chain polynomial $f_P(x)$ have only real zeros?
(Open.)



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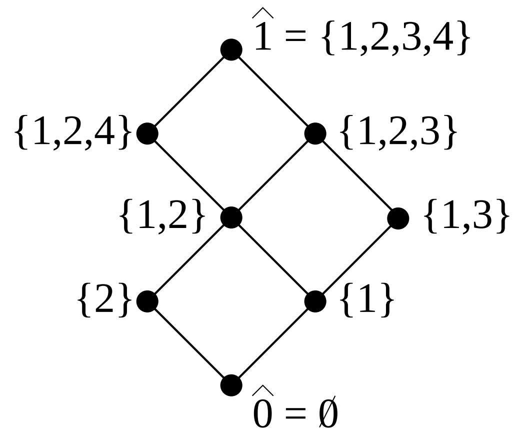


J(P)

Distributive lattices

For any poset P, define the *distributive lattice* $J(P)$ to be the poset of order ideals of P, ordered by inclusion.

Question. Which sequences $(a_0, \ldots, a_d)$ are f-vectors of distributive lattices?

(Open.)

The distributive lattice conjecture

Conjecture. (Neggers-Stanley) Let $J(P)$ be a finite distributive lattice. Then the chain polynomial $f_{J(P)}(x)$ has only real zeros.

Question. Is $f_{J(P)}$ always log-concave? (Open.)

Question. Is $f_{J(P)}$ always unimodal? (Open.)

h-vectors

The *h-vector* of a poset,

$$h_P = (h_0, \ldots, h_d),$$

is defined in terms of the f-vector,

$$h_P(x) = (1-x)^d f_P\left(\frac{x}{1-x}\right).$$

Example.

$$f_P(x) = 1 + 6x + 10x^2 + 5x^3.$$

$$\mapsto 5 + 10x + 6x^2 + x^3,$$

$$\mapsto 5 + 10(x-1) + 6(x-1)^2 + (x-1)^3$$

$$= 0 + x + 3x^2 + x^3,$$

$$\mapsto 1 + 3x + x^2 + 0x^3 = h_P(x).$$

Some h-vectors of distributive lattices

$$(1, 4, 4, 1)$$
$$(1, 6, 8, 2)$$
$$(1, 7, 14, 7, 1)$$
$$(1, 8, 11, 2)$$
$$(1, 8, 15, 8, 1)$$
$$(1, 9, 23, 13, 2)$$
$$(1, 10, 9)$$
$$(1, 10, 10)$$

Some f-vectors of posets

$$(1, 5, 7, 2)$$
$$(1, 5, 6, 1)$$
$$(1, 6, 9, 2)$$
$$(1, 7, 7, 1)$$
$$(1, 8, 11, 3)$$
$$(1, 9, 13, 3)$$
$$(1, 11, 20, 6)$$
$$(1, 13, 29, 12)$$

Theorem. (Stanley) *For each distributive lattice $J(P)$, there is a simplicial complex Γ such that $f_\Gamma = h_{J(P)}$.*

Question. For which posets P is there a poset Q such that $f_Q = h_{J(P)}$?
(Open.)

Theorem. *For each poset P which is a disjoint sum of chains, there is a poset Q such that $f_Q = h_{J(P)}$.*

Permutation statistics

A *permutation statistic* is a function
$$\phi : S_n \to \mathbb{N}.$$

The statistic *des* counts descents in a permutation. Dumont's statistic, *dmc*, counts the number of nonzero letters in code(π).

Example.

$$\begin{aligned} \pi &= 2\ 8\ 4\ 3\ 6\ 7\ 9\ 5\ 1 \\ \text{code}(\pi) &= 1\ 6\ 2\ 1\ 2\ 2\ 2\ 1\ 0 \end{aligned}$$

Positions $2, 3, 7, 8$ are descents in π. Thus, $\text{des}(\pi) = 4$.

The distinct nonzero letters in code(π) are $\{1, 2, 6\}$. Thus, $\text{dmc}(\pi) = 3$.

Distribution of des and dmc on $\mathbf{S_n}$

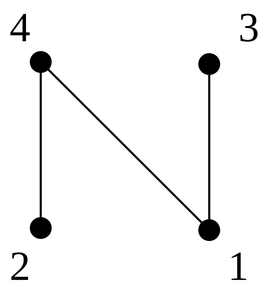
The permutation statistics des and dmc are called *Eulerian* because their distributions on S_n are given by the *Eulerian numbers*.

$$\begin{aligned} A(n, k+1) &= \#\{\pi \in S_n \,|\, \mathrm{des}(\pi) = k\}, \\ &= \#\{\pi \in S_n \,|\, \mathrm{dmc}(\pi) = k\}. \end{aligned}$$

$n \backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

A combinatorial interpretation of the h-vector

Theorem. (Stanley) *The h-vector* $h_{J(P)}$ *counts linear extensions of* P *by descent.*



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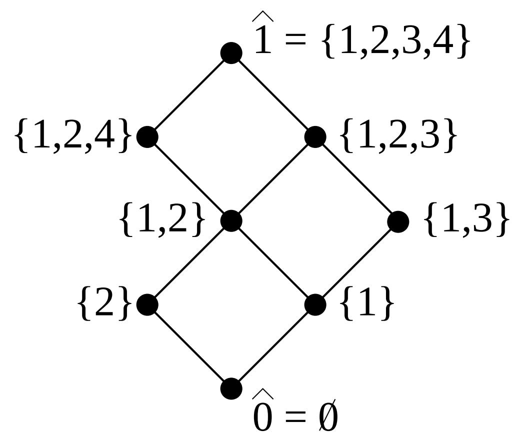
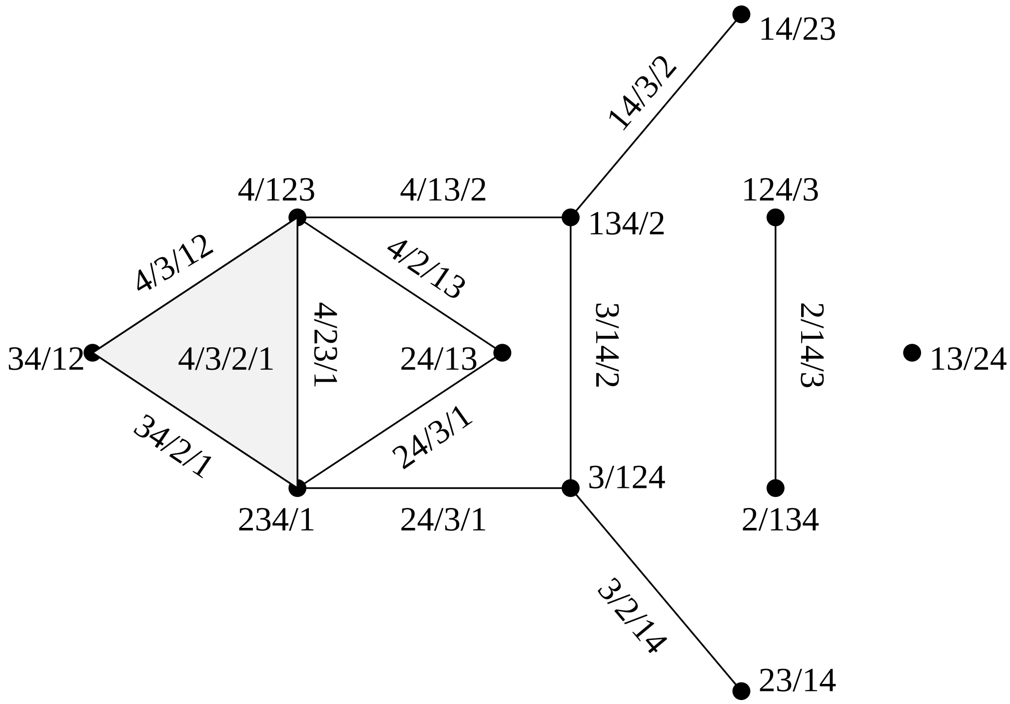


J(P)

$$f_{J(P)}(x) = (1+x)^2(1+6x+10x^2+5x^3),$$
$$h_{J(P)}(x) = 1+3x+x^2.$$

The linear extensions of P are

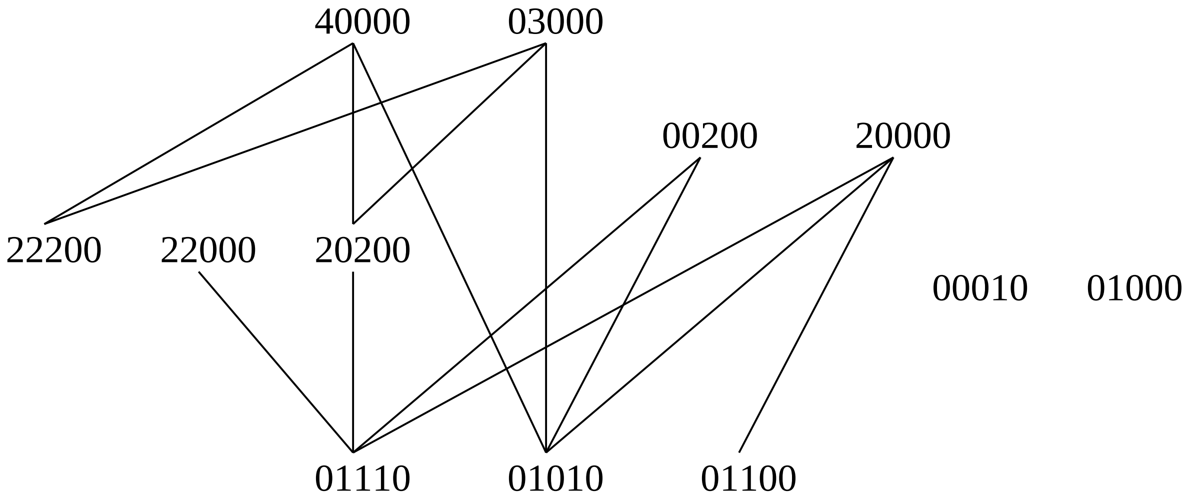
$$\{1234, 124/3, 13/24, 2/134, 2/14/3\}.$$



For any poset P we can construct a simplicial complex Γ satisfying $f_\Gamma = h_{J(P)}$ by associating a $(k-1)$-simplex to each linear extension of P which has k descents.

Example. If P is the antichain $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$, then Γ is the complex shown above.

$$f_\Gamma(x) = h_{J(P)}(x) = 1 + 11x + 11x^2 + x^3.$$



For any poset P which is a disjoint sum of chains, we can construct a poset Q satisfying $f_Q = h_{J(P)}$, by associating a k-element chain to each linear extension of P which has k letters in its code.

Example. If P is the sum of chains $\mathbf{2}+\mathbf{2}+\mathbf{1}$, then Q is the poset shown above.

$$f_Q(x) = h_{J(P)}(x) = 1 + 12x + 15x^2 + 2x^3.$$

Another conjecture

Conjecture. Let $J(P)$ be *any* finite distributive lattice. Then there is a poset Q such that $f_Q = h_{J(P)}$.

A strategy for proving the conjecture is to study *pairs* of permutation statistics.

A *linear extension* of a poset is a permutation which avoids certain forbidden *inversions*.

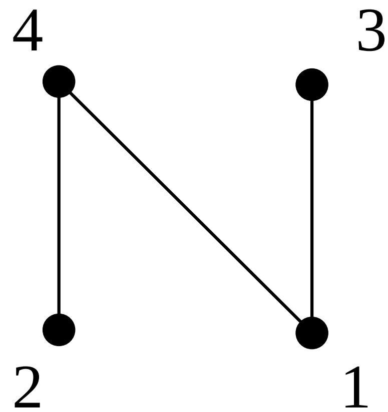
Inversions

Definition. An *inversion* in a permutation is a pair $(\pi_i > \pi_j)$ such that $i < j$.

$\textsc{inv}(\pi)$ counts the number of inversions in π.

Example. $\pi = 45123$ has six inversions:

$$(4,1),(4,2),(4,3),(5,1),(5,2),(5,3).$$



P

The linear extensions of P are

$$\{1234, 124/3, 13/24, 2/134, 2/14/3\}.$$

These are the permutations in S_4 avoiding the subsequences $41, 42, 31$.

Joint distributions

Theorem. (Foata, Zeilberger, Han) *The pairs of permutation statistics* $(\mathrm{des}, \mathrm{MAJ})$ *and* $(\mathrm{exc}, \mathrm{DEN})$ *are equally distributed on* S_n*.*

Theorem. (Foata) *The pairs of permutation statistics* $(\mathrm{des}, \mathrm{INV})$ *and* $(\mathrm{dmc}, \mathrm{MAJ})$ *are equally distributed on* S_n*.*

Question. Is there a natural Eulerian statistic *stat* such that the pairs $(\mathrm{des}, \mathrm{MAJ})$ and $(\mathrm{stat}, \mathrm{INV})$ are equally distributed on S_n?

Calculation of stc

Begin at the right of code(π), and moving left, circle the first letter which is at least 1, the next which is at least 2, etc. Set

$$\mathrm{stc}(\pi) = \text{ number of circles.}$$

Example.

$$\pi \quad = 4\ 6\ 2\ 3\ 5\ 1$$

$$\mathrm{code}(\pi) = 3\ 4\ 1\ 1\ 1\ 0.$$

$$\mathrm{stc}(462351) = 3.$$

Theorem. *The pairs of permutation statistics* $(\mathrm{stc}, \mathrm{INV})$ *and* $(\mathrm{des}, \mathrm{MAJ})$ *are equally distributed on* S_n.

Theorem. (Foata) *The pairs of permutation statistics* $(\mathrm{MAJ}, \mathrm{INV})$ *and* $(\mathrm{INV}, \mathrm{MAJ})$ *are equally distributed on* S_n.

Conjecture. The pairs of permutation statistics $(\mathrm{stc}, \mathrm{des})$ and $(\mathrm{des}, \mathrm{stc})$ are equally distributed on S_n.

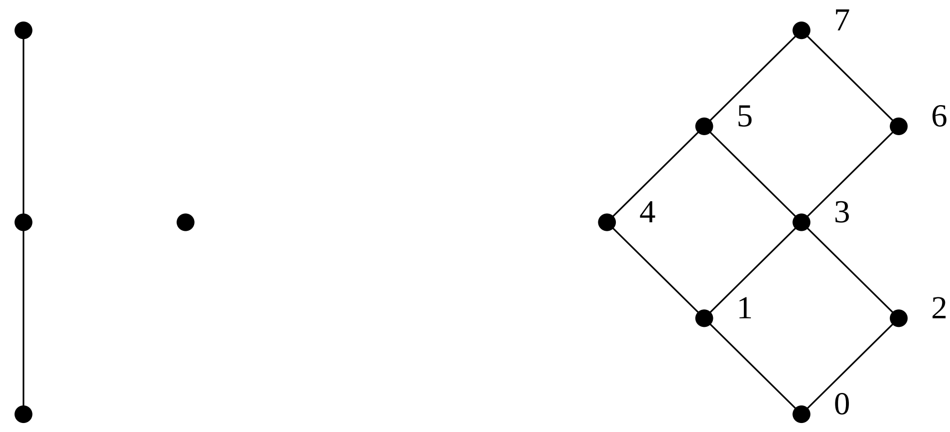
Question. Can the statistic stc be used to define a poset structure for S_n?

(Open.)

Silly question

Question. If $J(P)$ is a finite distributive lattice, can we find a finite poset Q such that $f_Q = h_{J(P)}$ *and* $f_Q(x)$ is known to have only real zeros?

This would prove two conjectures at once.



$(\mathbf{3}+\mathbf{1})$-free posets

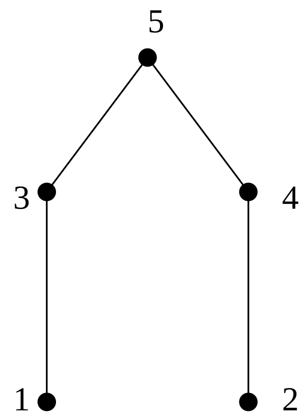
Call a poset $(\mathbf{3}+\mathbf{1})$-*free* if it contains no induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$.

The poset on the right above is not $(\mathbf{3}+\mathbf{1})$-free, because the subposet induced by elements $\{2,3,4,6\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$.

Question. Which sequences $(a_0, \ldots, a_d)$ are f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets?

(Open.)

Theorem. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the chain polynomial $f_P(x)$ has only real zeros.*



Antiadjacency matrices

Given a labelled poset P, we define its *antiadjacency matrix* $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 0, & \text{if } i <_P j, \\ 1, & \text{otherwise.} \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem. (Stanley) *The chain polynomial of a poset P is related to the antiadjacency matrix A by*

$$f_P(x) = \det(I + xA).$$

Theorem. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the antiadjacency matrix A corresponding to any labelling of P has only real eigenvalues.*

Let $J(P)$ be a distributive lattice with

$$h_{J(P)} = (h_0, h_1, \ldots, h_d).$$

Often there are many $(\mathbf{3}+\mathbf{1})$-free posets Q which satisfy $f_Q = h_{J(P)}$.

h_1	number of posets Q
1	1
2	1
3	1.33
4	2
5	2.9
6	3.53
7	6
8	8
9	13.43
10	18.51
11	25.97
12	40.36
13	58.53
14	98.53
15	111.97
16	237.72
17	263.30
18	507.75
19	493.10
20	1088.03