A GENERALIZATION OF DEODHAR'S DEFECT STATISTIC FOR IWAHORI–HECKE ALGEBRAS OF TYPE BC

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Outline

(1) Hyperoctahedral group - Coxeter group of type BC
(2) Bruhat order
(3) Type-BC Hecke algebra $H_n^{\mathsf{BC}}(q)$
(4) Star networks
(5) Path families and defects
(6) Combinatorial formula for multiplying in $H_n^{\mathsf{BC}}(q)$

The Hyperoctahedral group $\mathfrak{B}_n$

$\mathfrak{B}_n$ = permutations $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ of $\overline{n} \cdots \overline{2}\ \overline{1}\ 1\ 2 \cdots n$ with $w_{\overline{i}} = \overline{w_i}$. Generated by $s_0, s_1, \ldots, s_{n-1}$, subject to relations

$$\begin{aligned} {s_i}^2 &= e && \text{for } i = 0, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \quad i, j \geq 1, \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0 && \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Inherited action on words $a_{\overline{n}} \cdots a_{\overline{1}} a_1 \cdots a_n$ is

s_i swaps letters in positions $i, i+1$ and $\overline{i}, \overline{i+1}$,

s_0 swaps letters in positions $\overline{1}, 1$.

Ex: $s_2 s_1 s_0 (\overline{3}\overline{2}\overline{1}123) = s_2 s_1 (\overline{3}\overline{2}1\overline{1}23) = s_2(\overline{3}1\overline{2}2\overline{1}3) = 1\overline{3}\overline{2}23\overline{1}$.
Short one-line notation = right-half of one-line notation.

Reversals in $\mathfrak{B}_n$

The *Reversal* $s_{[a,b]}$ reverses letters $[a,b] := \{a,\ldots,b\} \setminus \{0\}$, where $a = \bar{b}$ $(b > 0)$ or $0 < a < b$.

Short one-line notation of $s_{[a,b]}$:

(1) (For $a = \bar{b}$) $\quad \bar{1} \cdots \bar{b} \cdot (b+1) \cdots n$

(2) (For $0 < a < b$) $\quad 1 \cdots (a-1) \cdot b \cdots a \cdot (b+1) \cdots n$

$s_{[a,b]}$ is the maximal element of the parabolic subgroup of $\mathfrak{B}_n$ generated by $\{s_{max\{0,a\}}, \ldots, s_{b-1}\}$.

Ex:

$s_{[2,4]}(\overline{54321}12345) = \overline{52341}14325.$

$s_{[\bar{4},4]}(\overline{54321}12345) = \bar{5}4321\overline{1234}5.$

Bruhat order on $\mathfrak{B}_n$

For $v, w \in \mathfrak{B}_n$, define $v \leq w$ if some (equivalently, every) reduced expression for w contains a reduced expression for v.

For a maximum element of a parabolic subgroup of $\mathfrak{B}_n$, define

$$\widetilde{C}_w(q) := \sum_{v \leq w} T_v$$

Ex: $\widetilde{C}_{s_{[1,3]}}(q) = T_e + T_{s_1} + T_{s_2} + T_{s_1 s_2} + T_{s_2 s_1} + T_{s_1 s_2 s_1}$.

$\widetilde{C}_w(q)$ belongs to a modified signless Kazhdan-Lusztig basis for $H_n^{\mathsf{BC}}(q)$.

The type-BC Hecke algebra $H_n^{\mathsf{BC}}(q)$

Define $H_n^{\mathsf{BC}}(q) = \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by $T_{s_0}, T_{s_1}, \ldots, T_{s_{n-1}}$, subject to relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 0, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_0}T_{s_1}T_{s_0}T_{s_1} &= T_{s_1}T_{s_0}T_{s_1}T_{s_0} && \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

For $w = s_{i_1} \cdots s_{i_\ell}$ reduced in $\mathfrak{B}_n$, define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$.

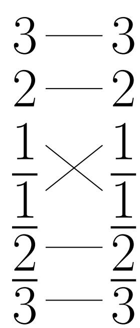
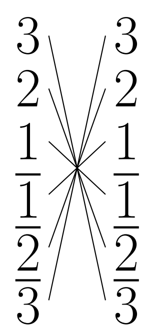
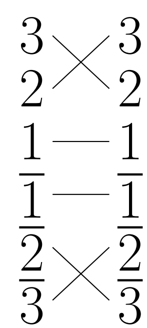
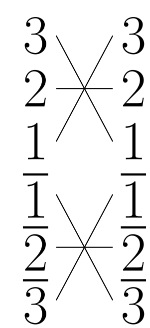
Natural basis: $\{T_w \,|\, w \in \mathfrak{B}_n\}$.

Fact: $H_n^{\mathsf{BC}}(1) \cong \mathbb{Z}[\mathfrak{B}_n]$.

Star networks

Define the type-BC *simple star network* $F_{s_{[a,b]}}$ to be the planar network corresponding to the reversal $s_{[a,b]} \in \mathfrak{B}_n$.

Ex: The reversals $s_{[\bar{1},1]}$, $s_{[\bar{3},3]}$, $s_{[2,3]}$, $s_{[1,3]}$ correspond to

$F_{s_{[\bar{1},1]}} =$ , $F_{s_{[\bar{3},3]}} =$ , $F_{s_{[2,3]}} =$ , $F_{s_{[1,3]}} =$ .

Define a type-BC *star network* $F = F_{s_{[a_1,b_1]}} \circ \cdots \circ F_{s_{[a_k,b_k]}}$ to be a concatenation of type-BC simple star networks.

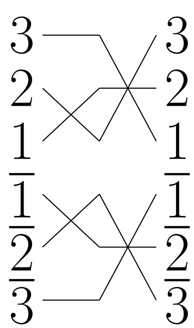
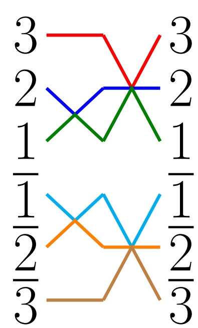
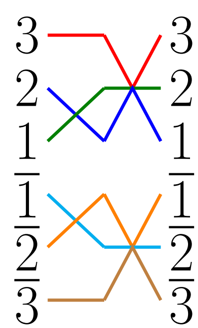
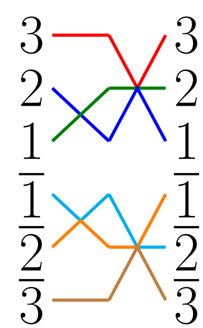
Path families in a star network F

Call $\pi = (\pi_{\overline{n}}, \ldots, \pi_n)$ a type-BC *path family in* F if:

(1) π_i is a path from source i on left to some sink on right,
(2) π covers all edges of F,
(3) π_i is a horizontal reflection of $\pi_{\overline{i}}$ for all $i = 1, \ldots, n$.

Then $\pi \in \Pi^{\mathsf{BC}}(F)$. Say type$(\pi) = w$ if π_i ends at sink w_i.

Ex: Consider the following path families in $F = F_{s_{[1,2]}} \circ F_{s_{[1,3]}}$

$F =$ , $\pi =$ , $\pi' =$ , $\sigma =$ 

π_e and π_{213} are type-BC, with type$(\pi) = e$, type$(\pi') = s_1$. But σ isn't a type-BC path family since σ_1 doesn't reflect $\sigma_{\overline{1}}$.

Defects of a path family π

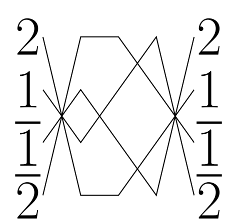
Let $F = F_{s_{[a_1,b_1]}} \circ \ldots \circ F_{s_{[a_t,b_t]}}$. Given $\pi \in \Pi^{\mathsf{BC}}(F)$, define a type-$\mathsf{BC}$ defect of π to be a triple (π_i, π_j, k) with

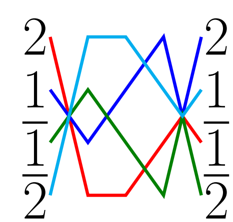
(1) $|i| \leq j$,

(2) π_i and π_j meet at an internal vertex of $F_{[c_k,d_k]}$ after having crossed an odd number of times.

Let $\mathrm{dfct}^{\mathsf{BC}}(\pi)$ denote the number of type-BC defects of π.

For example, consider the star network and path family

$F_{[\overline{2},2]} \circ F_{[\overline{1},1]} \circ F_{[1,2]} \circ F_{[\overline{2},2]} =$ 

, $\pi =$ 

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The defects of π are $(\pi_{\overline{1}}, \pi_1, 2)$, $(\pi_{\overline{1}}, \pi_2, 3)$, $(\pi_1, \pi_2, 4)$, $(\pi_{\overline{2}}, \pi_2, 4)$, and thus $\mathrm{dfct}^{\mathsf{BC}}(\pi) = 4$.

Graphical representations for $H_n^{\mathsf{BC}}(q)$

The set $\Pi^{\mathsf{BC}}(F)$ associates elements of $\mathbb{Z}[\mathfrak{B}_n]$ and $H_n^{\mathsf{BC}}(q)$ to F. We say that F *graphically represents*

$$\sum_{\pi \in \Pi^{\mathsf{BC}}(F)} \mathrm{type}(\pi)$$

as an element of $\mathbb{Z}[\mathfrak{B}_n]$ and

$$\sum_{\pi \in \Pi^{\mathsf{BC}}(F)} q^{\mathrm{dfct}^{\mathsf{BC}}(\pi)} T_{\mathrm{type}(\pi)}$$

as an element of $H_n^{\mathsf{BC}}(q)$.

Let $\Pi_{w,d}^{\mathsf{BC}}(F) = \{\pi \in \Pi^{\mathsf{BC}}(F) \,|\, \mathrm{type}(\pi) = w, \mathrm{dfct}^{\mathsf{BC}}(\pi) = d\}$.

Main Result

Thm: Fix a sequence $(s_{[a_1,b_1]}, \ldots, s_{[a_k,b_k]})$ of reversals in $\mathfrak{B}_n$ and define the type-BC star network $F = F_{s_{[a_1,b_1]}} \circ \ldots \circ F_{s_{[a_k,b_k]}}$. Then we have

$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_k,b_k]}}(q) = \sum_{w \in \mathfrak{B}_n} \sum_{d \geq 0} |\Pi^{\mathsf{BC}}_{w,d}(F)| q^d T_w.$$

This combinatorial formula for the coefficients in the natural expansion of such a product of Kazhdan-Lusztig basis elements yields a new procedure for multiplying in $H_n(q)$:

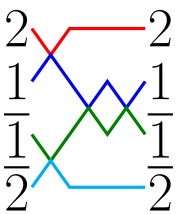
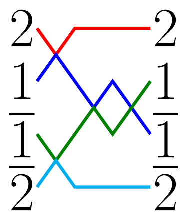
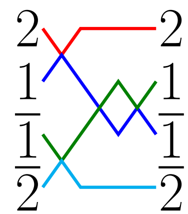
(1) Represent a product $\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_k,b_k]}}(q)$ by type-BC star network F.

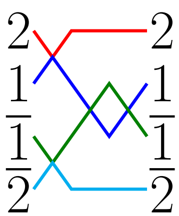
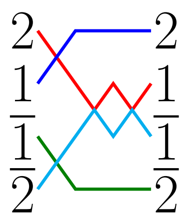
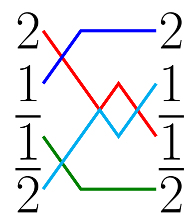
(2) Consider all possible path crossings to enumerate $\Pi^{\mathsf{BC}}(F)$.

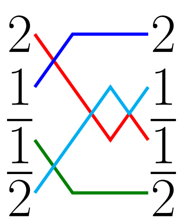
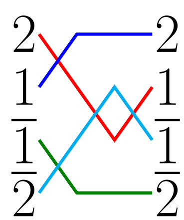
(3) Each $\pi \in \Pi^{\mathsf{BC}}_{w,d}(F)$ contributes $q^d T_w$ to the product.

Example multiplication in $H_n^{\mathsf{BC}}(q)$

$F = F_{s_1} \circ F_{s_0} \circ F_{s_0}$ represents $(T_e + T_{s_1})(T_e + T_{s_0})(T_e + T_{s_0})$.
There are 8 path families in $\Pi^{\mathsf{BC}}(F)$:

 $\in \Pi^{\mathsf{BC}}_{e,0}(F)$,  $\in \Pi^{\mathsf{BC}}_{s_0,0}(F)$,  $\in \Pi^{\mathsf{BC}}_{s_0,1}(F)$,

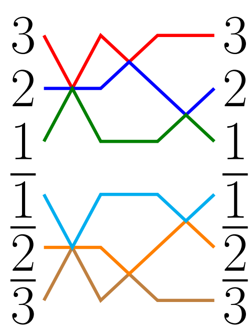
 $\in \Pi^{\mathsf{BC}}_{e,1}(F)$,  $\in \Pi^{\mathsf{BC}}_{s_1,0}(F)$,  $\in \Pi^{\mathsf{BC}}_{s_1 s_0,0}(F)$,

 $\in \Pi^{\mathsf{BC}}_{s_1 s_0,1}(F)$,  $\in \Pi^{\mathsf{BC}}_{s_1,1}(F)$.

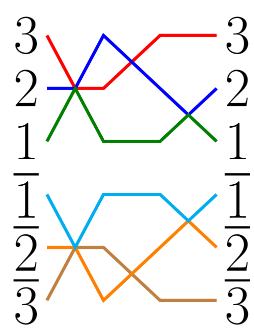
Combining each term gives $(1+q)(T_e + T_{s_0} + T_{s_1} + T_{s_1 s_0})$.

Example multiplication in $H_n^{\mathsf{BC}}(q)$

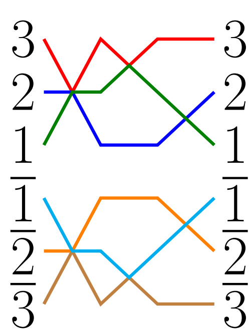
$F = F_{s_{[1,3]}} \circ F_{s_{[2,3]}} \circ F_{s_{[1,2]}}$ represents $\widetilde{C}_{s_{[1,3]}}(q)\widetilde{C}_{s_{[2,3]}}(q)\widetilde{C}_{s_{[1,2]}}(q)$.
To find the natural coefficient of T_e, observe



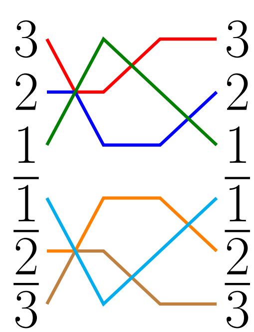
$\in \Pi_{e,0}^{\mathsf{BC}}(F)$,



$\in \Pi_{e,1}^{\mathsf{BC}}(F)$,



$\in \Pi_{e,1}^{\mathsf{BC}}(F)$,



$\in \Pi_{e,2}^{\mathsf{BC}}(F)$.

Therefore the T_e term has coefficient $(1 + 2q + q^2)$.