Expert Advice for Amateurs

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Online Appendix - Existence of Equilibria

The analysis in this section is performed under more general payoff functions. Without taking an explicit form, the payoffs of the expert and the decision maker, $U^e(a,\theta,b)$ and $U^d(a,\theta)$, are assumed to satisfy the following conditions: 1) $U^e(\cdot)$ and $U^d(\cdot)$ are twice continuously differentiable, 2) $U^i_{11}(\cdot)<0$, 3) $U^i_1(a,\cdot)=0$ for some $a\in\mathbb{R}$, 4) $U^i_{12}(\cdot)>0$, $i=e,d$, 5) $U^d(a,\theta)=U^e(a,\theta,0)$ for all (a,θ), and 6) $U^e_{13}(\cdot)>0$ everywhere. Also define

$$a(r,s)=\begin{cases}\underset{a'}{\text{argmax}}\int_r^s U^d(a',\theta)d\theta, & \text{if } r<s,\\ a^d(r), & \text{if } r=s.\end{cases}$$

The emergence of off-equilibrium beliefs in the amateur model, which cannot be eliminated by having all messages used, poses a challenge in the equilibrium characterization. A full characterization requires consideration of all possible off-equilibrium beliefs, which represent a very large set. In the following, I illustrate the issues and provide a specification of beliefs that, coupled with a mild condition on the expert's payoff, ensures the existence of partitional equilibria.

I first show that, for whatever beliefs after false advice, the expert never fully reveals his information. Intuitively, if the expert reveals θ, the amateur's own information will become useless, implying that the amateur will respond in exactly the same way as does the novice. Since a biased expert does not fully reveal to the novice, he also will not do so to the amateur.

Proposition 5. *There exists no separating equilibrium in the amateur model.*

Proof. If the expert in the amateur model fully reveals his information, for all $\theta\in\Theta$, θ induces $a^d(\theta)$ on all interval types. Accordingly, a type-θ expert's payoff is $\int_0^1 U^e(a^d(\theta),\theta,b)dt$. Suppose there exists a fully separating equilibrium. For any $\eta>0$, we can find a $\bar{\theta}<\eta$ that induces $a^d(\bar{\theta})$ in the equilibrium. Suppose $\bar{\theta}$ deviates by sending m' that in the equilibrium is reserved for $\theta'=\bar{\theta}+\epsilon$, where $\epsilon>0$ is such that $a^d(\bar{\theta})<a^d(\bar{\theta}+\epsilon)<a^e(\bar{\theta},b)$. The continuity of the payoff functions and that $U^d(a,\theta)\equiv U^e(a,\theta,0)$ guarantee that such an ϵ exists. Upon receiving m', all interval types with $t\notin(\bar{\theta},\bar{\theta}+\epsilon)$ will take action $a^d(\bar{\theta}+\epsilon)$, which, given the choice of ϵ and that $U^e_{11}(\cdot)<0$, is strictly preferred over $a^d(\bar{\theta})$ by $\bar{\theta}$. To the remaining types with $t\in(\bar{\theta},\bar{\theta}+\epsilon)$, all being low-interval types, m' is a false advice, and they take action under off-equilibrium beliefs $\psi(\theta|t_l)$. For any such beliefs, given that $U^d_{12}(\cdot)>0$, the set of induced actions taken by these interval types is bounded by $a^d(0)$ and $a^d(\bar{\theta}+\epsilon)$. Since $U^e_{11}(\cdot)<0$, we have $U^e(a^d(0), \bar{\theta}, b) \leq U^e(a', \bar{\theta}, b)$ for all $a' \in [a^d(0), a^d(\bar{\theta}+\epsilon)]$. Thus, the payoff for $\bar{\theta}$ to send m' is, regardless of the specification of $\psi(\theta|t_l)$, bounded below by

$$\text{(A.1)} \qquad \int_0^{\bar{\theta}} U^e(a^d(\bar{\theta}+\epsilon), \bar{\theta}, b)dt + \int_{\bar{\theta}}^{\bar{\theta}+\epsilon} U^e(a^d(0), \bar{\theta}, b)dt + \int_{\bar{\theta}+\epsilon}^{1} U^e(a^d(\bar{\theta}+\epsilon), \bar{\theta}, b)dt.$$

Subtracting $\bar{\theta}$'s equilibrium payoff from (A.1) gives:

$$\text{(A.2)} \qquad \begin{aligned} &\int_0^{\bar{\theta}} [U^e(a^d(\bar{\theta}+\epsilon), \bar{\theta}, b) - U^e(a^d(\bar{\theta}), \bar{\theta}, b)]dt + \int_{\bar{\theta}}^{\bar{\theta}+\epsilon} [U^e(a^d(0), \bar{\theta}, b) - U^e(a^d(\bar{\theta}), \bar{\theta}, b)]dt \\ &+ \int_{\bar{\theta}+\epsilon}^{1} [U^e(a^d(\bar{\theta}+\epsilon), \bar{\theta}, b) - U^e(a^d(\bar{\theta}), \bar{\theta}, b)]dt. \end{aligned}$$

We can choose η sufficiently small such that, for any ϵ that satisfies the above criterion of choosing the deviating message, the first and the third positive terms in (A.2) dominate the second negative terms for some $\bar{\theta} < \eta$. The strict incentive for some θ to deviate poses a contradiction to the existence of fully separating equilibrium.

□

In the CS model, when the indifference condition is satisfied under a *partitional strategy* of the expert (p.7-8) where $N \geq 2$, it follows that the following incentive-compatibility conditions are satisfied: 1) every boundary type θ_i will (weakly) prefer sending messages in M_i over any other messages; and 2) θ in the interior of every I_i—the *interior types*—will (strictly) prefer the same. The expert's strategy thus constitutes an informative partitional equilibrium. In the amateur model, even if the indifference condition holds under such strategy, in which[1]

$$\text{(A.3)} \qquad V^e(M_i, \theta_i, b) = V^e(M_{i+1}, \theta_i, b), \quad i = 1, \ldots, N-1, \ \theta_0 = 0 \text{ and } \theta_N = 1,$$

since the off-equilibrium beliefs may generate a "benefit of lying" under false advice the incentive-compatibility conditions is not necessarily satisfied.

Before illustrating how "benefit of lying" may arise, I pause to characterize the induced actions of the amateur under a partitional strategy. I distinguish between two types of induced actions, effectively induced and ineffectively induced. An action a is effectively induced by m if the amateur updates her beliefs $\mu(\theta|m, t_s)$ using Bayes' rule and there exists $\theta \in \Theta$ such that, in her maximization problem of which a is the solution, $\mu(\theta|m, t_s) \neq \phi(\theta|t_s)$.

Lemma 4. *A high-interval type t_h takes effectively induced actions if and only if she receives*

1. *substituting advice: her threshold $t \in I_i$, $i = 1, \ldots, N-1$, and the expert reveals that $\theta \in I_j$, $i < j \leq N$; or*
2. *complementary advice: her threshold $t \in I_i$, $i = 1, \ldots, N-1$, and the expert reveals that $\theta \in I_i$,*

and she takes ineffectively induced actions if and only if she receives

[1] With a slight abuse of notations, I use $V^e(M_i, \theta_i, b)$ to stand for $V^e(m, \theta_i, b)$ for all $m \in M_i$.

3. redundant advice: her threshold $t \in I_N$, and the expert reveals that $\theta \in I_N$; or

4. false advice: her threshold $t \in I_i$, $i = 2, \dots, N$, and expert reveals that $\theta \in I_k$, $1 \le k < i$.

A low-interval type t_l takes effectively induced actions if and only if she receives

5. substituting advice: her threshold $t \in I_i$, $i = 2, \dots, N$, and the expert reveals that $\theta \in I_k$, $1 \le k < i$; or

6. complementary advice: her threshold $t \in I_i$, $i = 2, \dots, N$, and the expert reveals that $\theta \in I_i$,

and she takes ineffectively induced actions if and only if she receives

7. redundant advice: her threshold $t \in I_1$, and the expert reveals that $\theta \in I_1$;

8. false advice: her threshold $t \in I_i$, $i = 1, \dots, N-1$, and the expert reveals that $\theta \in I_j$, $i < j \le N$.

***Proof**.* I prove the cases for high-interval types; the cases for low-interval types are similar. Consider first Conditions 1 and 2. Suppose t_h with $t \in I_i$, $i = 1, \dots, N-1$, receives m indicating that $\theta \in I_j$. For $j > i$, t_h's updated beliefs under Bayes' rule are $\mu(\theta|m, t_h) = 1/(\theta_j - \theta_{j-1})$ for $\theta \in (\theta_{j-1}, \theta_j]$ and zero elsewhere. For $j = i$, t_h's updated beliefs are $\mu(\theta|m, t_h) = 1/(\theta_i - t)$ for $\theta \in [t, \theta_i]$ and zero elsewhere. In both cases, there exists $\theta \in [0, 1]$ such that $\mu(\theta|m, t_h) \neq \phi(\theta|t_h)$, and the resulting actions are effectively induced. This proves the sufficiencies. The necessities is proved by contrapositive. Suppose t_h with $t \in I_i$, $i = 1, \dots, N-1$, receives m indicating that $\theta \in I_j$, $j < i$. Note that then $\Theta_\sigma(m) \cap t_h = \emptyset$; Bayes' rule cannot be applied, and the resulting action cannot be effectively induced. Finally, suppose t_h with $t \in I_N$ receives m indicating that $\theta \in I_N$. Her updated beliefs are $\mu(\theta|m, t_h) = 1/(1-t)$ for $[t, 1]$ and zero elsewhere. This is equivalent to $\phi(\theta|t_h)$ for all $\theta \in [0, 1]$, and the resulting action cannot be effectively induced. Since there are only two types of actions, effectively induced and ineffectively induced, and they are mutually exclusive, the sufficiencies (necessities) in Conditions 3 and 4 for ineffectively induced actions follow from the above necessities (sufficiencies) for effectively induced actions.

□

Using Lemma 4, the profile of actions effectively induced on t_h, $t \in I_i$, $i = 1, \dots, N-1$, is

$$\rho(m, t_h) = \begin{cases} a(t, \theta_i), & \text{for } m \in M_i \text{ (complementary advice)}, \\ a(\theta_{j-1}, \theta_j), & \text{for } m \in M_j,\ i < j \le N \text{ (substituting advice)}; \end{cases}$$

and that induced on t_l, $t \in I_i$, $i = 2, \dots, N$ is

$$\rho(m, t_l) = \begin{cases} a(\theta_{i-1}, t), & \text{for } m \in M_i \text{ (complementary advice)}, \\ a(\theta_{k-1}, \theta_k), & \text{for } m \in M_k,\ 1 \le k < i \text{ (substituting advice)}. \end{cases}$$

The profile of actions ineffectively induced by redundant advice is $\rho(m, t_h) = a(t, 1)$ and $\rho(m, t_l) = a(0, t)$. The profile of actions induced by false advice depends on the off-equilibrium beliefs. To illustrate the "benefit of lying," suppose the indifference condition holds with $N = 3$ in a proposed equilibrium. Consider boundary type θ_1 (Figure 3).[2] If θ_1 is the true state, all high-interval types will have $t \leq \theta_1$ and all low-interval types $t > \theta_1$. Suppose θ_1 sends $m \in M_1$, indicating that $\theta \in [0, \theta_1]$. Then, all high-interval types will take action $a(t, \theta_1)$, $t \in [0, \theta_1]$, and all low-interval types will take action $a(0, \theta_1)$ (the second line in Figure 3). Given that θ_1 satisfies the indifference condition, he will be indifferent between inducing these actions and those induced by $m \in M_2$, which are $a(\theta_1, \theta_2)$ and $a(\theta_1, t)$, $t \in (\theta_1, \theta_2]$ (not shown in the figure).

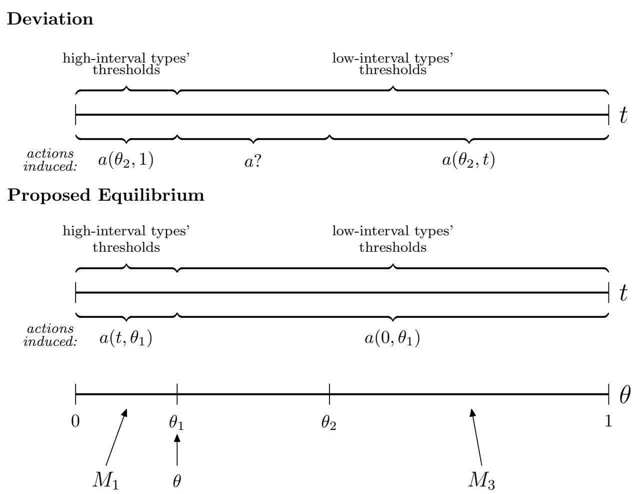

Figure 1: Incentives for Deviations

Now, suppose θ_1 lies by sending $m \in M_3$, indicating that $\theta \in (\theta_2, 1]$. All high-interval types will take $a(\theta_2, 1)$ and all low-interval types with $t \in (\theta_2, 1]$ will take $a(\theta_2, t)$ (the first line in Figure 3). These interval types cannot detect the lie and are effectively induced to take these actions. And to θ_1 these actions are less favorable than those induced by M_1 (or M_2). To the low-interval types with $t \in (\theta_1, \theta_2]$, the advice can be detected as false. Without any restriction, one can come up with off-equilibrium beliefs so that the ineffectively induced actions taken by these low-interval types will be closer to θ_1's ideal than is $a(0, \theta_1)$, the effectively induced action taken by these types in the proposed equilibrium. This creates a benefit of lying, which is absent in the CS model. It is conceivable, especially in equilibria with more steps, that such benefit of lying may outweigh the cost of inducing less favorable actions. What equilibria may emerge in situations of this sort require *ad hoc* and detailed specifications of beliefs.

The following proposition states, however, that there is a set of off-equilibrium beliefs that, together with a mild restriction on the expert's payoff, ensures the sufficiency of the indifference condition for the existence of partitional equilibria. Denote ψ to be the set of off-equilibrium beliefs of all interval types: $\psi = \bigcup_{\{t,s\} \in T \times \{l,h\}} \psi(\theta | t_s)$.

Proposition 6. *There exists a set of off-equilibrium beliefs ψ^* such that, provided $U^e_{12}(\cdot)$ is sufficiently large, the boundary types $\{\theta_i\}_{i=1}^{N-1}$ that satisfy* (A.3) *always constitute an equilibrium.*

[2] There could be a benefit of lying for boundary types whenever $N \geq 3$. However, for the interior types, such benefit also arises for $N = 2$. Thus, the indifference condition is not always sufficient even for two-step equilibria. Indeed, in the CS model, incentive compatibility for the interior types is a consequence of that for the boundary types. As will be discussed below, this is also not true in the amateur model.

Proof. I first construct ψ^* and state the cases of the expert's payoff $V^e(m,\theta,b)$ under ψ^*. I then show that, if $\psi=\psi^*$ and $U^e_{12}(\cdot)$ is sufficiently large, then (A.3) is sufficient for the following to always hold: for all $\theta \in [\theta_{i-1},\theta_i]$,

$$V^e(M_i,\theta,b)=\max_j V^e(M_j,\theta,b),\ i,j=1,\dots,N. \tag{A.4}$$

The set of off-equilibrium beliefs ψ^* is constructed as follows. Suppose there exists a monotone solution, $\{\theta_1,\dots,\theta_{N-1}\}\subset(0,1)$, to (A.3). If a high-interval type t_h with $t\in(\theta_i,\theta_{i+1}]$, $i=0,\dots,N-1$, receives a false advice, her beliefs are that θ is distributed on $[t,\theta_{i+1}]$ with density $1/(\theta_{i+1}-t)$ and zero elsewhere; if a low-interval type t_l with $t\in(\theta_i,\theta_{i+1}]$ receives a false advice, her beliefs are that θ is distributed on (θ_i,t) with density $1/(t-\theta_i)$ and zero elsewhere. Then, when $\theta\in[\theta_{i-1},\theta_i]$ sends $m\in M_i$ under the partitional strategy and deviates from it by sending $m\in M_g$, $g\neq i$, the profile of his expected payoff will be

$$V^e(m,\theta,b)=\begin{cases}\displaystyle\int_0^{\theta_{k-1}}U^e(a(\theta_{k-1},\theta_k),\theta,b)dt \\ \displaystyle\qquad+\sum_{r=k-1}^{i-1}\int_{\theta_r}^{\theta_{r+1}}U^e(a(t,\theta_{r+1}),\theta,b)dt \\ \displaystyle\qquad+\int_{\theta_i}^{\theta}U^e(a(t,\theta_{i+1}),\theta,b)dt & \text{if } m\in M_k,\ 1\le k<i, \\ \displaystyle\qquad+\int_{\theta}^{1}U^e(a(\theta_{k-1},\theta_k),\theta,b)dt, \\ \displaystyle\int_0^{\theta_{i-1}}U^e(a(\theta_{i-1},\theta_i),\theta,b)dt \\ \displaystyle\qquad+\int_{\theta_{i-1}}^{\theta}U^e(a(t,\theta_i),\theta,b)dt \\ \displaystyle\qquad+\int_{\theta}^{\theta_i}U^e(a(\theta_{i-1},t),\theta,b)dt & \text{if } m\in M_i, \\ \displaystyle\qquad+\int_{\theta_i}^{1}U^e(a(\theta_{i-1},\theta_i),\theta,b)dt, \\ \displaystyle\int_0^{\theta}U^e(a(\theta_{j-1},\theta_j),\theta,b)dt \\ \displaystyle\qquad+\int_{\theta}^{\theta_i}U^e(a(\theta_{i-1},t),\theta,b)dt \\ \displaystyle\qquad+\sum_{r=i}^{j-1}\int_{\theta_r}^{\theta_{r+1}}U^e(a(\theta_r,t),\theta,b)dt & \text{if } m\in M_j,\ i<j\le N. \\ \displaystyle\qquad+\int_{\theta_j}^{1}U^e(a(\theta_{j-1},\theta_j),\theta,b)dt, \end{cases} \tag{A.5}$$

Using the second and the third cases in (A.5), the expected payoff for θ_i to send $m\in M_i$ and

$m \in M_{i+1}$ are, respectively,

$$\text{(A.6)} \qquad \int_0^{\theta_{i-1}} U^e(a(\theta_{i-1},\theta_i),\theta_i,b)dt + \int_{\theta_{i-1}}^{\theta_i} U^e(a(t,\theta_i),\theta_i,b)dt + \int_{\theta_i}^{1} U^e(a(\theta_{i-1},\theta_i),\theta_i,b)dt$$

$$\text{(A.7)} \qquad \int_0^{\theta_i} U^e(a(\theta_i,\theta_{i+1}),\theta_i,b)dt + \int_{\theta_i}^{\theta_{i+1}} U^e(a(\theta_i,t),\theta_i,b)dt + \int_{\theta_{i+1}}^{1} U^e(a(\theta_i,\theta_{i+1}),\theta_i,b)dt.$$

Thus, the indifference condition (A.3) becomes the following second-order difference equation:

$$\text{(A.8)} \qquad \begin{aligned} V(\theta_{i-1},\theta_i,\theta_{i+1},b) = &\int_0^{\theta_{i-1}} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_{i-1},\theta_i),\theta_i,b)]dt \\ &+ \int_{\theta_{i-1}}^{\theta_i} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(t,\theta_i),\theta_i,b)]dt \\ &+ \int_{\theta_i}^{\theta_{i+1}} [U^e(a(\theta_i,t),\theta_i,b) - U^e(a(\theta_{i-1},\theta_i),\theta_i,b)]dt \\ &+ \int_{\theta_{i+1}}^{1} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_{i-1},\theta_i),\theta_i,b)]dt = 0, \end{aligned}$$

$i = 1,\ldots,N-1$, $\theta_0 = 0$, $\theta_N = 1$. Suppose there is a strictly increasing partition, $\theta_0,\ldots,\theta_i$, that satisfies (A.8). That $U^e_{11}(\cdot) < 0$, $a(\cdot,\cdot)$ is strictly increasing in its arguments, and the continuity of $V(\theta_{i-1},\theta_i,\theta',b)$ in θ' ensure that there exists a unique $\theta_{i+1} > \theta_i$ that satisfies (A.8).

Turning to incentive compatibility, I begin by showing that (A.4) holds for θ_i, $i = 1,\ldots,N-1$, that satisfy (A.3). If $N = 2$, there exists no other set of messages that θ_i can send, and (A.4) is satisfied vacuously. So, consider $N \geq 3$. Suppose θ_i sends message $m \in M_{i+n}$, $2 \leq n \leq N-i$. Then, from the third case in (A.5) his expected payoff is

$$\text{(A.9)} \qquad \begin{aligned} &\int_0^{\theta_i} U^e(a(\theta_{i+n-1},\theta_{i+n}),\theta_i,b)dt + \sum_{r=i}^{i+n-2} \int_{\theta_r}^{\theta_{r+1}} U^e(a(\theta_r,t),\theta_i,b)dt \\ &+ \int_{\theta_{i+n-1}}^{\theta_{i+n}} U^e(a(\theta_{i+n-1},t),\theta_i,b)dt + \int_{\theta_{i+n}}^{1} U^e(a(\theta_{i+n-1},\theta_{i+n}),\theta_i,b)dt. \end{aligned}$$

Subtracting (A.9) from (A.7), we have

$$\begin{aligned} D_3 = &\int_0^{\theta_i} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_{i+n-1},\theta_{i+n}),\theta_i,b)]dt \\ &+ \int_{\theta_i}^{\theta_{i+1}} [U^e(a(\theta_i,t),\theta_i,b) - U^e(a(\theta_i,t),\theta_i,b)]dt \\ &+ \sum_{r=i+1}^{i+n-2} \int_{\theta_r}^{\theta_{r+1}} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_r,t),\theta_i,b)]dt \\ &+ \int_{\theta_{i+n-1}}^{\theta_{i+n}} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_{i+n-1},t),\theta_i,b)]dt \\ &+ \int_{\theta_{i+n}}^{1} [U^e(a(\theta_i,\theta_{i+1}),\theta_i,b) - U^e(a(\theta_{i+n-1},\theta_{i+n}),\theta_i,b)]dt. \end{aligned}$$

Note that (A.8) implies that the expert's ideal action $a^e(\theta_i, b) \in (a(\theta_{i-1}, \theta_i), a(\theta_i, \theta_{i+1}))$. Since $a(\theta_{i+n-1}, \theta_{i+n}) > a(\theta_i, \theta_{i+1})$, $a(\theta_{i+n-1}, t) > a(\theta_i, \theta_{i+1})$ for $t \in (\theta_{i+n-1}, \theta_{i+n})$, and $a(\theta_j, t) > a(\theta_i, \theta_{i+1})$ for $t \in (\theta_j, \theta_{j+1})$, $j = i+1, \ldots, i+n-2$, given $U^e_{11}(\cdot) < 0$ and the maximum of $U^e(a, \theta_i, b)$ is achieved for $a \in (a(\theta_{i-1}, \theta_i), a(\theta_i, \theta_{i+1}))$, the first, third, fourth and fifth terms are positive. Also, the second term vanishes. Thus, $D_3 > 0$. Next, suppose θ_i sends $m \in M_{i-\eta}$, $1 \leq \eta \leq i-1$. From the first case in (A.5), his expected payoff is

$$\begin{aligned}
&\int_0^{\theta_{i-\eta-1}} U^e(a(\theta_{i-\eta-1}, \theta_{i-\eta}), \theta_i, b)dt + \int_{\theta_{i-\eta-1}}^{\theta_{i-\eta}} U^e(a(t, \theta_{i-\eta}), \theta_i, b)dt \\
&\quad + \sum_{r=i-\eta}^{i-1} \int_{\theta_r}^{\theta_{r+1}} U^e(a(t, \theta_{r+1}), \theta_i, b)dt + \int_{\theta_i}^{1} U^e(a(\theta_{i-\eta-1}, \theta_{i-\eta}), \theta_i, b)dt.
\end{aligned} \tag{A.10}$$

Subtracting (A.10) from (A.6), we have

$$\begin{aligned}
D_4 = &\int_0^{\theta_{i-\eta-1}} [U^e(a(\theta_{i-1}, \theta_i), \theta_i, b) - U^e(a(\theta_{i-\eta-1}, \theta_{i-\eta}), \theta_i, b)]dt \\
&+ \int_{\theta_{i-\eta-1}}^{\theta_{i-\eta}} [U^e(a(\theta_{i-1}, \theta_i), \theta_i, b) - U^e(a(t, \theta_{i-\eta}), \theta_i, b)]dt \\
&+ \sum_{r=i-\eta}^{i-2} \int_{\theta_r}^{\theta_{r+1}} [U^e(a(\theta_{i-1}, \theta_i), \theta_i, b) - U^e(a(t, \theta_{r+1}), \theta_i, b)]dt \\
&+ \int_{\theta_{i-1}}^{\theta_i} [U^e(a(t, \theta_i), \theta_i, b) - U^e(a(t, \theta_i), \theta_i, b)]dt \\
&+ \int_{\theta_i}^{1} [U^e(a(\theta_{i-1}, \theta_i), \theta_i, b) - U^e(a(\theta_{i-\eta-1}, \theta_{i-\eta}), \theta_i, b)]dt.
\end{aligned}$$

Similar to the above, since $a(\theta_{i-1}, \theta_i) > a(\theta_{i-\eta-1}, \theta_{i-\eta})$, $a(\theta_{i-1}, \theta_i) > a(t, \theta_{i-\eta})$ for $t \in (\theta_{i-\eta-1}, \theta_{i-\eta})$, and $a(\theta_{i-1}, \theta_i) > a(t, \theta_{j+1})$, for $t \in (\theta_j, \theta_{j+1})$, $j = i-\eta, \ldots, i-2$, the first, second, third and fifth terms are positive, while the fourth term vanishes. Thus, $D_4 > 0$. That $D_3 > 0$ and $D_4 > 0$ imply that (A.4) holds for θ_i, $i = 1, \ldots, N-1$.

I show next that given (A.8) and for sufficiently large $U^e_{12}(\cdot)$, all $\theta \in (\theta_{i-1}, \theta_i)$ prefer sending $m \in M_i$ over $m \in M_{i+1}$, and all $\theta \in (\theta_i, \theta_{i+1})$ prefer sending $m \in M_{i+1}$ over $m \in M_i$, $i = 1, \ldots, N-1$, so that (A.4) holds for all interior θ. Consider an arbitrary $\theta \in (\theta_{i-1}, \theta_i)$. From the third case in (A.5), his expected payoff from sending $m \in M_{i+1}$ is

$$\begin{aligned}
&\int_0^{\theta} U^e(a(\theta_i, \theta_{i+1}), \theta_i, b)dt + \int_{\theta}^{\theta_i} U^e(a(\theta_{i-1}, t), \theta_i, b)dt \\
&\quad + \int_{\theta_i}^{\theta_{i+1}} U^e(a(\theta_i, t), \theta_i, b)dt + \int_{\theta_{i+1}}^{1} U^e(a(\theta_i, \theta_{i+1}), \theta_i, b)dt.
\end{aligned} \tag{A.11}$$

Subtracting his expected payoff from sending $m \in M_i$ in (A.5) from (A.11), we have

$$
\begin{aligned}
D_5 = & \int_0^{\theta_{i-1}} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt \\
& + \int_{\theta_{i-1}}^{\theta} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(t, \theta_i), \theta, b)]dt \\
& + \int_{\theta}^{\theta_i} [U^e(a(\theta_{i-1}, t), \theta, b) - U^e(a(\theta_{i-1}, t), \theta, b)]dt \\
& + \int_{\theta_i}^{\theta_{i+1}} [U^e(a(\theta_i, t), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt \\
& + \int_{\theta_{i+1}}^{1} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt.
\end{aligned} \tag{A.12}
$$

Differentiating D_5 with respect to θ gives

$$
\begin{aligned}
\frac{\partial D_5}{\partial \theta} = & \int_0^{\theta_{i-1}} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta_{i-1}}^{\theta} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(t, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta_i}^{\theta_{i+1}} \frac{\partial [U^e(a(\theta_i, t), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta_{i+1}}^{1} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta, \theta_i), \theta, b)]\theta.
\end{aligned}
$$

Since $a(\theta_i, \theta_{i+1}) > a(\theta_{i-1}, \theta_i)$, $a(\theta_i, \theta_{i+1}) > a(t, \theta_i)$ for $t \in (\theta_{i-1}, \theta)$, and $a(\theta_i, t) > a(\theta_{i-1}, \theta_i)$ for $t \in (\theta_i, \theta_{i+1})$, $U^e_{12}(\cdot) > 0$ implies that the first four terms are positive and the last term is negative; when θ decreases from θ_i, there are negative effects on D_5 from the first four term and a positive effect from the last term. However, for a sufficiently large $U^e_{12}(\cdot)$ at θ, the negative effects outweigh the positive. A sufficiently large $U^e_{12}(\cdot)$ then ensures, given (A.8), $D_5 \leq 0$ for θ. Consider next an arbitrary $\theta \in (\theta_i, \theta_{i+1})$. From the first case in (A.5), his expected payoff from sending $m \in M_i$ is

$$
\begin{aligned}
& \int_0^{\theta_{i-1}} U^e(a(\theta_{i-1}, \theta_i), \theta, b)dt + \int_{\theta_{i-1}}^{\theta_i} U^e(a(t, \theta_i), \theta, b)dt \\
& + \int_{\theta_i}^{\theta} U^e(a(t, \theta_{i+1}), \theta, b)dt + \int_{\theta}^{1} U^e(a(\theta_{i-1}, \theta_i), \theta, b)dt.
\end{aligned} \tag{A.13}
$$

Subtracting (A.13) from the expected payoff from sending $m \in M_{i+1}$ in (A.5), we have

$$
\begin{aligned}
D_6 = & \int_0^{\theta_{i-1}} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt \\
& + \int_{\theta_{i-1}}^{\theta_i} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(t, \theta_i), \theta, b)]dt \\
& + \int_{\theta_i}^{\theta} [U^e(a(t, \theta_{i+1}), \theta, b) - U^e(a(t, \theta_{i+1}), \theta, b)]dt \\
& + \int_{\theta}^{\theta_{i+1}} [U^e(a(\theta_i, t), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt \\
& + \int_{\theta_{i+1}}^{1} [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]dt.
\end{aligned}
\tag{A.14}
$$

Differentiating D_6 with respect to θ gives

$$
\begin{aligned}
\frac{\partial D_6}{\partial \theta} = & \int_0^{\theta_{i-1}} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta_{i-1}}^{\theta_i} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(t, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta}^{\theta_{i+1}} \frac{\partial [U^e(a(\theta_i, t), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& + \int_{\theta_{i+1}}^{1} \frac{\partial [U^e(a(\theta_i, \theta_{i+1}), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]}{\partial \theta} dt \\
& - [U^e(a(\theta_i, \theta), \theta, b) - U^e(a(\theta_{i-1}, \theta_i), \theta, b)]\theta.
\end{aligned}
$$

Since $a(\theta_i, \theta_{i+1}) > a(\theta_{i-1}, \theta_i)$, $a(\theta_i, \theta_{i+1}) > a(t, \theta_i)$ for $t \in (\theta_{i-1}, \theta)$, and $a(\theta_i, t) > a(\theta_{i-1}, \theta_i)$ for $t \in (\theta_i, \theta_{i+1})$, $U^e_{12}(\cdot) > 0$ implies that the first four terms are positive. While the last term is negative, similar to the above, for a sufficiently large $U^e_{12}(\cdot)$ at θ, the positive effects on D_6 from the first four terms outweigh the negative effect from the last term; (A.8) then implies $D_6 \geq 0$ at θ. When $U^e_{12}(\cdot)$ is sufficiently large for all interior types, (A.4) holds for all of them.

□

The off-equilibrium beliefs specified in the proof—that for $t \in (\theta_i, \theta_{i+1}]$, t_h and t_l believe that θ is uniformly distributed on, respectively, $[t, \theta_{i+1}]$ and (θ_i, t)—are sufficient for the incentive compatibility condition to hold for the boundary types. However, since the interior types induce a set of actions not induced by the boundary types, a sufficiently large $U^e_{12}(\cdot)$ is called into the picture to ensure that incentive compatibility also holds for them. Figure 4 illustrates the rationale with an example of two-step equilibrium.

Given that the indifference condition holds, the boundary type θ_1's expected payoff from the profile of actions $a(0, \theta_1)$ and $a(t, \theta_1)$, $t \in [0, \theta_1]$, is the same from that from $a(\theta_1, 1)$ and $a(\theta_1, t)$, $t \in (\theta_1, 1]$ (the two upper lines). Consider the actions induced when the interior type θ sends messages in M_1 and M_2. If we compare the profile of actions in the lower pair of lines with those in the upper pair, we can see that they are the same except for $t \in (\theta, \theta_1]$. While θ_1 gives no

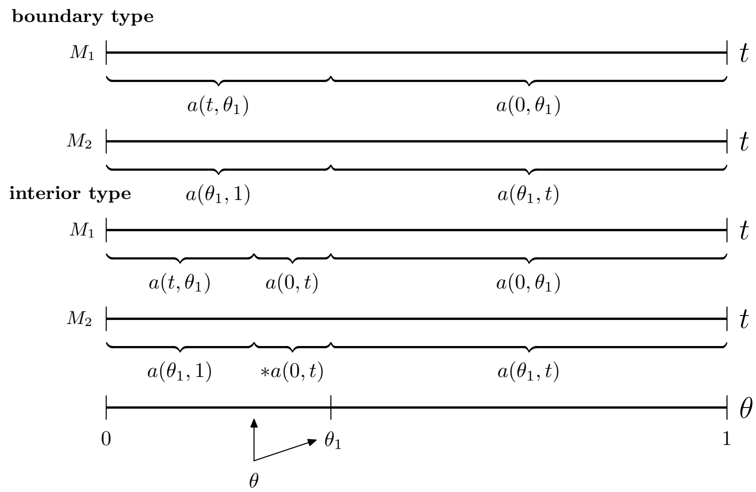

Figure 2: Actions Induced by Boundary and Interior Types

false advice when he sends messages in either M_1 and M_2, there is one when θ sends $m \in M_2$. The specification of ψ^*, which allows incentive compatibility to hold for θ_1, (ineffectively) induce the action $a(0, t)$ (with asterisk) for the interior type if he sends messages in M_2, which is the same as the action effectively induced by $m \in M_1$.

If we could fix the profile of actions, that $U_{12}^e(\cdot) > 0$ would have guaranteed that $\theta < \theta_1$ strictly prefers to send messages in M_1 over M_2. However, when the expert's type changes, the profile of actions also changes, and, insofar as the actions taken by $t \in (\theta, \theta_1]$ are concerned, θ is indifferent between M_1 and M_2. Thus, we have to ensure that, overall, θ prefers M_1 enough for $t \notin (\theta, \theta_1]$ so that even with the indifference for $t \in (\theta, \theta_1]$ the incentive compatibility still holds. For this, a sufficiently large $U_{12}^e(\cdot)$ is required. A large $U_{12}^e(\cdot)$ means that the ideal action of a higher θ is sufficiently higher than that of a lower θ. This additional restriction is nothing but a strengthening of the already existing sorting condition.