Volatility Smiles and Yield Frowns

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Abstract

A volatility smile relates some measure of an option's implied volatility (IV) eg. annualized variance rate, to some measure of the option's moneyness, eg. strike minus forward. The term smile is used because the graph is typically convex. A yield curve relates the yield to maturity (YTM) on a bond to the bond's time to maturity. When the yield curve is concave, which is typically the case, it is natural to to analogously refer to this curve as a yield frown. In this paper, we show that the IV smile and the YTM frown are both due to randomness in future IV's and future YTM's respectively. We develop simple models for the risk-neutral dynamics of IV's and YTM's that lead to a quadratic curve for each.

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“There is a fundamental similarity between the role of interest rates in the pricing of bonds and the role of volatility in the pricing of index options.” – Emanuel Derman et. al. (Investing in Volatility).

“This note explores the analogy between the dynamics of the interest rate term structure and the implied volatility surface of a stock.” – Rogers and Tehranchi.

1 Introduction

The interaction between option pricing and bond pricing has a long history. A footnote in Merton[9]'s pathbreaking paper on the pricing of stock options develops the first bond pricing model by assuming that the short interest rate follows arithmetic Brownian motion. It was quickly realized that the role of an underlying asset's volatility in pricing an option written on this asset is similar to the role of an interest rate in pricing a bond. Nowadays, implied volatilities are used as a quotation convention for OTC FX options, just as bond yields have long been used to quote bonds. Historical data on bond yields and implied variance rates has shown that both time series are mean-reverting and non-negative. This allows dynamical models developed for one rate to be used to describe the other.

There has been a fruitful interplay between models developed to price bonds and models developed to price options. For example, Black, Derman, and Toy[3] developed a binomial tree for interest rates which was designed to be consistent with an initially given yield curve. This motivated Derman and Kani[6] to develop a binomial tree for stock prices, which was designed to be consistent with an initially given implied volatility surface. To take another example, Cox Ingersoll and Ross[5] develop a model for pricing bonds, which uses a mean-reverting square root process to describe the evolution of the short interest rate. In the CIR model, an explicit formula for pricing a zero coupon bond is available despite the complicated path dependence that enters the pricing operator. Afterwards, Heston[8] developed a model for pricing stock options, which uses the same mean-reverting square root process to describe the evolution of the short variance rate of the underlying stock. In Heston's model, one has a semi-explicit formula for the price of a stock option, which has contributed greatly to the model's popularity.

To take another type of example, Dybvig Ingersoll, and Ross [7] proved that long interest rates can never fall. Afterwards, Rogers and Tehranchi[10] prove that long implied volatility can never fall as well.

To take yet another example, Armerin, Jensen, and Bjork[1] show that if the term structure of yields is required to be flat at all calendar times, then the absence of arbitrage forces this flat term structure to also be constant as calendar time evolves. Put another way, an initially flat yield curve cannot move by parallel shifts. Ross conjectures the parallel result for the graph of implied volatility against strike. Rogers and Tehranchi[10] confirm this conjecture under a mild regulatory condition.

The purpose of this paper is to shed some new light on the surprising connection between interest rates and volatility. More precisely, we explore the connection between the continuously compounded yield on a zero coupon bond and the implied volatility of a European swaption. A yield curve relates the yield to maturity on a bond to the bond's time to maturity. A volatility smile[1] relates some measure of the implied volatility of an option's underlying eg. annualized variance rate, to some measure of the option's moneyness. The main objective of this paper is to draw a precise mathematical connection between a volatility smile and a yield curve. In this paper, we find particular measures of implied volatility and moneyness of a European swaption, so that the resulting volatility smile is analogous to the yield curve for zero coupon bonds. The relation is used to develop arbitrage-free curves in both cases.

At any calendar time $t \geq 0$, we define the *yield curve* as the graph of the continuously compounded yield $y_t(\tau)$ of a zero coupon bond maturing at some fixed maturity date $T \geq t$ against the term $\tau \equiv T - t$. At any calendar time $t \geq 0$, we use the term *volatility smile* whenever some measure of the time t volatility of the forward swap rate underlying a swaption is graphed against some measure of the swaption's time t moneyness at a fixed term and tenor.

For the volatility measure, we use a normal implied variance rate arising from the Bachelier model, rather than a lognormal implied variance rate arising from the more popular Black model. We have illustrated our option results with swaptions because the standard quotation convention in that market uses normal volatilities rather than lognormal volatilities.

In a Bachelier setting, one can consider at least four measures of moneyness:

1. strike rate $K \in \mathbb{R}$
2. $K - F_t$, where $F_t \in \mathbb{R}$ is the time t forward swap rate at the fixed term and tenor.
3. $F_t - K$ measures the in-the-moneyness of a call while $K - F_t$ measures the in-the-moneyness of a put.
4. $\frac{F_t - K}{\eta_t}$ where η_t is also the annualized implied standard deviation of the terminal forward swap rate (at term τ and at the moneyness level used to define it).

The analogy between the yield curve and the volatility smile will be most transparent when the last measure of moneyness is used in conjunction with a normal annualized variance rate to define the volatility smile. We will show that this last measure of moneyness can be interpreted as the number of annualized standard deviations that the forward swap rate exceeds the strike rate.

When we draw the analogy between yield curves and volatility smiles, we consider different models for each. We first consider yields in a very simple model where the short interest rate is constant over time. The absence of arbitrage forces the yield curve to be flat and constant over time.

We first consider normal variance rates in a slightly more realistic model where the short interest rate is stochastic. As a result, the yield curve will not be flat and it will evolve randomly over calendar time. However, the model in which we first consider normal variance rates is one in which the instantaneous normal variance rate of each forward swap rate does not vary over time. As a result, the absence of arbitrage forces the forward swap rate (normal) volatility cube to be constant over time. When we fix term and tenor of the forward swap rate to specific values, the resulting volatility smile is flat and constant over calendar time.

[1]The term smile does not imply that the graph is required to be convex, although it often is.

To summarize, the yield curve is flat and constant over calendar time in a model of constant short interest rates. Similarly, the volatility smile is flat and constant over calendar time in a model of constant short normal variance rates. We next draw an analogy between the yield curve and the volatility smile when short interest rates are allowed to be stochastic in the first model, and when short variance rates are allowed to be stochastic in the second model. We posit continuous dynamics under a martingale measure on yields in the stochastic short interest rate model and we posit continuous dynamics under a martingale measure on normal implied volatilities (square root of implied variance rates) in the stochastic short variance rate model. These continuous dynamics allow us to develop arbitrage-free curves in both cases. When we restrict the continuous dynamics to parallel shifts, we find that the yield curve and the volatility smile are both quadratic. As a result, the absence of arbitrage and non-trivial parallel shifts imply non-flat curves which evolve randomly over time.

An overview of this paper is as follows. The next section considers benchmark models i.e. models used to define the concept of yield to maturity or implied variance rate. In the bond's benchmark model, the yield curve is flat and constant over time, while in the swaption's benchmark model, the volatility smile is flat and constant over time. The following section considers market models, i.e. models which describe how yields and implied volatilities evolve over time under an appropriately chosen martingale measure. The final section summarizes the paper and suggests extensions. An appendix contains a technical result concerning the effect of repeatedly applying a particular differential-integro operator to a function.

2 Benchmark Models

In this section, we describe two different models, each of which have different purposes. A constant short interest rate model is used to define the yield to maturity of a bond. Similarly, a constant short normal volatility model is used to define the normal implied volatility of a swaption. In the next section, we describe two additional models. A market model of stochastic yields is used to build an arbitrage-free yield curve. Analogously, a market model of stochastic implied volatilities is used to build an arbitrage-free volatility smile. In this section, we will draw analogies between the first two models which both have a single parameter. In the next section, we will also draw an analogy between the second two models, which impose continuous random dynamics on this single parameter.

2.1 Zero Coupon Bonds and Money Market Account

In this subsection, we state some assumptions which are in force when the objective is to value bonds. We assume that at least one zero coupon bond of a fixed maturity date $T \geq 0$ trades. When we use the word bond in the sequel, it can always assumed to be a zero-coupon default-free bond. When we use the word $T-$bond in the sequel, the T refers to the bond's maturity date. Ruling out arbitrage in the $T-$bond market implies that the value b of the $T-$bond at any time $t \in [0, T]$ must be positive. Buying a $T-$bond with zero or negative price is an obvious arbitrage opportunity. We do not assume that money can be stored costlessly, so a $T-$bond price above one does not necessarily imply an arbitrage.

We also assume that there exists a money market account (MMA) which is default-free. Let $r_s \in \mathbb{R}$ be the possibly random interest rate which the MMA pays at every future date $s \geq 0$. An initial dollar investment in the MMA results in $e^{\int_0^t r_s ds}$ dollars at time t, for all $t \geq 0$.

2.1.1 Constant Interest Rate Bond Model

In this subsection, we assume that the short interest rate is correctly known at time 0 to be constant through time at $r \in \mathbb{R}$. Hence, an initial dollar investment in the MMA results in e^{rt} dollars at time t, for all $t \geq 0$. Ruling out arbitrage between the MMA and a $T-$ bond implies that the value b of the $T-$bond at any time $t \in [0, T]$ must be:

$$b(r, t; T) = e^{-r(T-t)}, \qquad r \in \mathbb{R}, t \in [0, T], T \geq 0. \tag{1}$$

If at some time $t \in [0, T]$ the $T-$ bond is priced at some level a_t below $e^{-r(T-t)}$, then the arbitrage involves purchasing the $T-$ bond at time t for a_t dollars and holding it to its maturity date T. The arbitrage is completed by shorting $e^{-r(T-t)}$ worth of the MMA at time t and holding it to the bond's maturity date[2] T. At the strategy entry time $t \in [0, T]$, the arbitrageur pockets $e^{-r(T-t)} - a_t > 0$ dollars and at the strategy exit time T, the dollar received from the expiring long bond can be used to close the short MMA position. If at some time $t \in [0, T]$ the $T-$ bond is instead priced at some level a_t above $e^{-r(T-t)}$, then the above positions are reversed so that the arbitrage profit at the arbitrage entry time t is $a_t - e^{-r(T-t)} > 0$ dollars. Assuming zero price impact, both arbitrages can be scaled up to produce infinite profits.

An easy way to determine whether a given bond is mis-priced by our simple model is to compute the yield-to-maturity (YTM) of the bond. Given that the time t market price of some $T-$bond is known to be some positive number $b_t(T)$ for T fixed, then whether or not interest rates are stochastic, the yield-to-maturity of this $T-$bond is defined as the solution to the equation:

$$b_t(T) = B(y_t(\tau), \tau), \qquad t \geq 0, \tag{2}$$

where $\tau \equiv T - t$ and:

$$B(r, \tau) \equiv b(r, t; T) = e^{-r\tau}, \qquad r \in \mathbb{R}, \tau \geq 0. \tag{3}$$

The function B is decreasing in its first argument so the solution y of (2) always exists for any positive b. In fact, from (2) and (3), we have the following well known explicit formula for the yield:

$$y_t(\tau) \equiv -\frac{\ln b_t(T)}{\tau}, \qquad t \geq 0, \tau \geq 0, \tag{4}$$

where $\tau \equiv T - t$. For any $t \geq 0$, the graph of the $T-$bond's yield $y_t(\tau)$ against its term $\tau \equiv T - t$ is called the yield curve at time t. If a market maker quotes any real yield curve $y_t(\tau), \tau > 0$ directly, then bond prices for any maturity date can be calculated by:

$$b_t(T) = e^{-y_t(\tau)\tau}, \qquad t \geq 0, \tau \geq 0. \tag{5}$$

[2]This can be rephrased as rolling over short-term borrowing.

Since the resulting bond prices are positive, they are arbitrage-free[3].

We now return to the world of a constant short interest rate. Replacing the market bond price $b_t(T)$ in the YTM definition (4) with the model bond value $b(r,t;T) = e^{-r(T-t)}$ implies that the only yield curve which is free of our model-based arbitrage is:

$$y_t(T) = r, \qquad r \in \mathbb{R}, t \in [0,T], T \geq 0. \tag{6}$$

Thus, our model-based arbitrage-free yield curve is flat at r and stays constant as calendar time advances. If two yields of different maturity differ on a fixed date, then either the model is correct and hence there is an arbitrage, or else the model is incorrect, so one needs to at least introduce deterministic short rates. Similarly, if two observations of the yield of some fixed term, eg. 10 years, differ over two different calendar times, then the model is incorrect, so one needs to at least introduce deterministic short rates.

Before discussing swaptions, it will be helpul to describe this simple bond math using the language of ordinary differential equations. In our constant short rate model, the arbitrage-free bond price $b(r,t;T)$ is the unique solution to the following terminal value problem:

$$\frac{db(r,t;T)}{dt} - rb(r,t;T) = 0, \qquad r \in \mathbb{R}, t \in [0,T], T \geq 0, \tag{7}$$

subject to:

$$b(r,T;T) = 1, \qquad r \in \mathbb{R}, T \geq 0. \tag{8}$$

The time homogeneity of the model implies that the pricing time t and the bond's maturity date T enter only through their difference $\tau \equiv T - t$. As a result, the function $B(r,\tau) \equiv b(r,t;T)$ is the unique solution to the following *initial* value problem:

$$\frac{dB(r,\tau)}{d\tau} = -rB(r,\tau), \qquad r \in \mathbb{R}, \tau \geq 0, \tag{9}$$

subject to:

$$B(r,0) = 1, \qquad r \in \mathbb{R}. \tag{10}$$

The solution to this initial value problem is simply:

$$B(r,\tau) = e^{-r\tau}, \qquad r \in \mathbb{R}, \tau \geq 0. \tag{11}$$

2.2 Payer Swaptions and Forward Starting Annuity

A payer interest rate swap (IRS) forces it owner to pay a fixed interest rate and receive a floating interest rate until the swap matures at some fixed date U. The fixed interest rate is determined at the inception of the swap and is chosen so that the payer IRS has zero cost of entry. This fixed interest rate is called the swap rate.

[3]Recall that we do not assume that money can be stored costlessly, so implied negative forward rates do not necessarily imply arbitrage.

A forward payer interest rate swap entered into at t forces its owner into a payer IRS at a fixed future date $T \geq t$. The fixed interest rate paid between the fixed dates T and U is determined at the inception date t and is chosen so that the forward payer IRS has zero cost of entry at t. This fixed interest rate is called the forward swap rate. To value a forward starting swap after inception, let $A_t(T, U)$ be the spot price at time $t \in [0, T]$ of a forward starting annuity whose unit cash flows begin at T and end at $U > T$. Let $K(T, U)$ be the fixed rate determined at time 0. Let $F_t(T, U)$ be the forward swap rate at time $t \in [0, T]$ for an interest rate swap beginning at T and ending at $U > T$. In what follows, we fix both T and U so we drop the arguments of A, K, and F. Then the value at time t of the forward starting swap is $A_t(F_t - K)$ dollars.

A payer swaption gives its owner the right to enter into a payer IRS that begins on the same date T that the swaption matures. Let $U > T$ be the maturity date of the underlying swap. The time span $T - t$ is called the term of the swaption, while the time span $U - T$ is called the tenor of the swaption.

For a payer swaption, the payer swap received upon exercise forces its owner to pay a fixed rate determined at inception and receive the floating interest rate. Of course, the owner of the payer swaption will only exercise at T if the value at T of the floating rate receipts over (T, U) exceeds the value at T of the fixed rate payments over (T, U). The payoff at T from a payer swaption is $A_T(F_T - K)^+$ dollars. If the forward starting annuity (FSA) is treated as a numeraire, then the payoff in FSA's is simply $(F_T - K)^+$. Hence, a payer swaption is essentially a European call option written on the forward swap rate.

2.2.1 Constant Normal Volatility Swaption Model

The last subsection assumed that the short interest rate is constant through time. In this subsection, we instead assume that the short interest rate is stochastic. However, the instantaneous normal volatility of a forward swap rate will be constant through time. We call this model the Bachelier[2] swaption pricing model. Under the forward swap measure $\mathbb{Q}^a$ corresponding to that term and tenor, the forward swap rate dynamics are assumed to be:

$$dF_t = \eta dW_t, \qquad t \geq 0, \tag{12}$$

where the constant η is the instantaneous normal volatility of the forward swap rate. Here, W is a $\mathbb{Q}^a$ standard Brownian motion.

Let c denote the arbitrage-free value of a call swaption, measured in FSA's. Let $c(F, t; K, T, \eta)$ be the the arbitrage-free call swaption valuation function in our constant normal volatility model. This function is the unique solution to the following terminal value problem:

$$\frac{\partial}{\partial t} c(F, t; K, T, \eta) + \frac{\eta^2}{2} \frac{\partial^2}{\partial F^2} c(F, t; K, T, \eta) = 0, \qquad F \in \mathbb{R}, t \in [0, T], \tag{13}$$

subject to:

$$c(F, T; K, T, \eta) = (F - K)^+, \qquad F \in \mathbb{R}. \tag{14}$$

The time homogeneity of the model implies that the pricing time t and the swaption's maturity date T enter only through their difference $\tau \equiv T - t$. Similarly, the spatial homogeneity of the model implies that the underlying forward rate F and the strike rate K enter only through their difference $x \equiv F - K$. As a result, the function $C(\eta, x, \tau) \equiv c(F, t; K, T, \eta)$ is the unique solution to the following *initial* value problem:

$$\frac{\partial}{\partial \tau} C(\eta, x, \tau) = \frac{\eta^2}{2} \frac{\partial^2}{\partial x^2} C(\eta, x, \tau), \qquad x \in \mathbb{R}, \tau \geq 0, \tag{15}$$

subject to:

$$C(\eta, x, \tau) = x^+, \qquad x \in \mathbb{R}. \tag{16}$$

The closed form solution due to Bachelier is:

$$C(\eta, x, \tau) = \eta N' \left(\frac{x}{\eta \sqrt{\tau}} \right) + x N \left(\frac{x}{\eta \sqrt{\tau}} \right), \tag{17}$$

where $N(z) \equiv \int\limits_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the well known standard normal distribution function.

Now allow variance rates to be stochastic and let $c_t(K, T, U)$ be the market price in FSA's of a K, T, U-payer swaption at time $t \in [0, T]$. Recall $\eta_t(m), m = x/\eta$ denotes the normal implied volatility of a $K, T, U-$ payer swaption at time $t \in [0, T]$. Since the swaption maturity date T and the underlying swap maturity date U are both considered fixed, our notation $\eta_t(m)$ supresses the dependence of the implied volatility η on the term $\tau \equiv T - t$ and on the tenor $U - T$. When a call price of a fixed strike rate $K \in \mathbb{R}$, swaption maturity date T, and swap maturity date U is known at time t, then the normal implied variance rate $\eta_t^2(m), m = (F_t - K)/\eta$, is defined as the solution to the equation:

$$c_t(K, T, U) = C\left(\sqrt{\eta_t^2(m)}, X_t, \tau \right), \qquad K \in \mathbb{R}, U \geq T \geq t \geq 0, \tag{18}$$

where $X_t \equiv F_t - K$. Since the function $C(\eta, x, \tau)$ given in (17) is increasing in η^2, the inverse map relating η^2 to C exists, but is not explicit. However, if we substitute out the second argument $x = F - K$ in favor of our moneyness measure $m = x/\eta$, then we get a new formula for the call value:

$$\hat{C}(\eta, m, \tau) = \eta \left[N' \left(\frac{m}{\sqrt{\tau}} \right) + m N \left(\frac{m}{\sqrt{\tau}} \right) \right], \qquad \eta > 0, m \in \mathbb{R}, \tau > 0. \tag{19}$$

Suppose we alternatively define the normal implied variance rate $\hat{\eta}_t^2(m), m = (F_t - K)/\eta$, as the solution to the equation:

$$c_t(K, T, U) = \hat{C}\left(\sqrt{\hat{\eta}_t^2(m)}, m, \tau \right), \qquad K \in \mathbb{R}, U \geq T \geq t \geq 0. \tag{20}$$

Then like yields, there is an explicit formula for this alternative normal implied variance rate:

$$\hat{\eta}_t^2(m) = \left(\frac{c_t(K, T, U)}{N' \left(\frac{m}{\sqrt{\tau}} \right) + m N \left(\frac{m}{\sqrt{\tau}} \right)} \right)^2, \qquad t \geq 0, K \in \mathbb{R}, U \geq T \geq t \geq 0. \tag{21}$$

Replacing the market price $c_t(K,T,U)$ in (18) with the Bachelier model value $\eta N'\left(\frac{X_t}{\eta\sqrt{\tau}}\right) + X_t N\left(\frac{X_t}{\eta\sqrt{\tau}}\right)$ implies that in the Bachelier model, the usual normal implied volatility smile is flat and constant over time:

$$\eta_t^2(m) = \eta^2, \qquad t \geq 0, m \in \mathbb{R}. \tag{22}$$

Hence, the normal implied variance rate is a useful tool for detecting violations of the Bachelier model. If options at two different moneyness levels differ in terms of their implied variance rates, then either the Bachelier model dynamics are correct and there is an arbitrage, or else the Bachelier model dynamics are wrong. If two observations of the implied volatility η of some fixed moneyness, eg. at-the-money differ over two different calendar times, then the Bachelier model is wrong. If the Bachelier model dynamics are wrong, then one needs to either add path-dependence, jumps in F, or make the short variance rate η^2 stochastic by for example allowing it to depend on the forward swap rate F.

If a market maker quotes a positive implied volatility smile $\eta_t^2(m), m \in \mathbb{R}$ directly, then swaption prices for any moneyness can be calculated by:

$$c_t(K,T,U) = \sqrt{\eta_t^2(m)}\left[N'\left(\frac{m}{\sqrt{\tau}}\right) + mN\left(\frac{m}{\sqrt{\tau}}\right)\right], \qquad t \geq 0, K \in \mathbb{R}, T \geq t, \tag{23}$$

where $K = F_t - m\sqrt{\eta_t^2(m)}$ and $T = t + \tau$. These prices are free of simple arbitrages that involve just one strike rate and maturity date. However, they are not necessarily free of arbitrages that involve more than one strike rate or more than one maturity date. This is a well known drawback of quoting by implied volatility. A similar problem would arise with quoting yields if one were to demand that forward interest rates be non-negative.

2.3 Comparing Benchmark Models

It is interesting to compare the Bachelier IVP (15) and (16) governing swaption values with the bond IVP (9) and (10) governing bond prices. We repeat both initial value problems here:

$$\frac{d}{d\tau}B(r,\tau) = -rB(r,\tau), \qquad \tau \geq 0, \qquad \text{s.t.} \qquad B(r,0) = 1, \tag{24}$$

$$\frac{\partial}{\partial\tau}C(\eta,x,\tau) = \frac{\eta^2}{2}\frac{\partial^2}{\partial x^2}C(\eta,x,\tau), x \in \mathbb{R}, \tau \geq 0, \qquad \text{s.t.} \qquad C(\eta,x,0) = x^+, x \in \mathbb{R}. \tag{25}$$

Let interest and variance rates be random. For $t \geq 0$, the yield curve $\{y_t(T), T \geq t\}$ solves[4]:

$$\frac{\partial}{\partial\tau}B(y_t(\tau),\tau) = -y_t(T)B(y_t(T),\tau), \qquad \text{s.t.} \qquad B(y_T(T),0) = 1 \qquad \tau = T - t \geq 0, \tag{26}$$

while for each fixed term $\tau > 0$, the implied normal volatility smile $\{\eta_t^2(m), m = x/\eta\}$ solves:

$$\frac{\partial}{\partial\tau}C(\eta_t(m), F_t - K, \tau) = \frac{\eta_t^2(m)}{2}\frac{\partial^2}{\partial x^2}C(\eta_t(m), F_t - K, \tau), \qquad \text{s.t.} \qquad C(\eta_T(m), x, 0) = x^+, \qquad m, x \in \mathbb{R}. \tag{27}$$

[4]Notice that the LHS of (26) is now a partial derivative, not a total derivative. The τ in $y_t(\tau)$ is held fixed.

We refer to the function $\frac{\partial^2}{\partial x^2}C(\eta,x,\tau)$ in (27) as the call's gamma although one should remember that C is the call's value in FSA's, not dollars. On the RHS of (26), the yield is multiplied by the negative of the bond price, while on the RHS of (27), the implied variance rate is multiplied by half the call's gamma. We claim that for each fixed $\tau > 0$, the call's gamma $\frac{\partial^2}{\partial x^2}C(\eta,x,\tau)$ plays the same role in linking the implied normal variance rate η^2 to the moneyness measure $m = x/\eta$ as the bond's pricing function $B(r,\tau)$ does in relating the bond's yield to maturity y to the bond's term τ. The reason for this claim will become clear in the next section.

For now, we observe that the bond pricing function $B(r,\tau)$ is a fundamental solution of the operator $\mathcal{D}_\tau + r\mathcal{I}$ since (24) implies that:

$$(\mathcal{D}_\tau + r\mathcal{I})B(r,\tau) = \delta(\tau), \qquad r \in \mathbb{R}, \tau \in \mathbb{R}, \tag{28}$$

where $\delta(\tau)$ is a Dirac delta function in the τ variable. Let $\Gamma(\eta,x,\tau) \equiv \frac{\partial^2}{\partial x^2}C(\eta,x,\tau)$ be the function relating the call swaption's gamma to normal volatility η, excess $x = F - K$, and term τ. Differentiating the Bachelier call formula (17) twice w.r.t. x implies that:

$$\Gamma(\eta,x,\tau) = \frac{N'\left(\frac{x}{\eta\sqrt{\tau}}\right)}{\eta\sqrt{\tau}}, \qquad \eta > 0, x \in \mathbb{R}, \tau > 0. \tag{29}$$

We observe that the call's gamma function $\Gamma(\eta,x,\tau)$ is also a fundamental solution, but for a different operator than $B(r,\tau)$. Differentiating the IVP (25) twice w.r.t. x implies that $\Gamma(\eta,x,\tau)$ is a fundamental solution of the operator $\mathcal{D}_\tau - \frac{\eta^2}{2}\mathcal{D}_{xx}$ since:

$$\left(\mathcal{D}_\tau - \frac{\eta^2}{2}\mathcal{D}_{xx}\right)\Gamma(\eta,x,\tau) = \delta(\tau)\delta(x), \qquad \eta \in \mathbb{R}, x \in \mathbb{R}, \tau \in \mathbb{R}, \tag{30}$$

where $\delta(x)$ is a Dirac delta function in the x variable.

The introduction of stochastic interest rates and stochastic volatilities will change the operators that these fundamental solutions solve. However, when we set the coefficients in these new operators to yield to maturity and to the normal implied variance rate respectively, then the functions $B(y,\tau) = e^{-y\tau}$ and $\Gamma(\eta,x,\tau) = \frac{N'\left(\frac{x}{\eta\sqrt{\tau}}\right)}{\eta\sqrt{\tau}}$ will also be fundamental solutions for the new problems.

Besides being fundamental solutions, the two functions B and Γ each arise in the Rodrigues formula and inner product for Laguerre polynomials and for the probabilists' Hermite polynomials. The Laguerre polynomials, usually denoted $L_0, L_1, \ldots$ are a polynomial sequence which may be defined by the following Rodrigues formula:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n, \qquad n = 0, 1, \ldots \tag{31}$$

The Laguerre polynomials are orthogonal with respect to the following inner product:

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx. \tag{32}$$

Similarly, the "probabilists' Hermite polynomials" denoted $He_0, He_1, \ldots$ are given by the following Rodrigues formula:

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1, \qquad n = 0, 1, \ldots \tag{33}$$

The probabilists' Hermite polynomials are orthogonal with respect to the following inner product:

$$\int_{-\infty}^{\infty} He_m(x) He_n(x)\, e^{-\frac{x^2}{2}}\, \mathrm{d}x = \sqrt{2\pi} n! \delta_{nm} \qquad m, n = 0, 1, \ldots. \tag{34}$$

One can also define generalized Laguerre polynomials and physicists' Hermite polynomials H_n. In fact, the generalized Laguerre polynomials are related to the physicists' Hermite polynomials:

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2), \qquad n = 0, 1, \ldots \tag{35}$$

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{(1/2)}(x^2), \qquad n = 0, 1, \ldots \tag{36}$$

Thus, the connection between the benchmark models for bonds and for options is not as surprising as it may first seem.

3 Market Models

3.1 Market Model for Yields

In common with the subsection with the Bachelier swaption pricing model, we assume that the short interest rate is stochastic. However, our focus is on pricing bonds, not swaptions. We assume that the market gives us initial yields of zero coupon bonds at a finite number of maturities. The objective is to connect the dots, so as to produce a full yield curve.

We assume no arbitrage and that $\mathbb{P}$ is the real world probability measure. Let $\mathbb{Q}$ be the martingale measure equivalent to $\mathbb{P}$, which arises when the MMA is taken to be the numeraire. Suppose that under $\mathbb{Q}$, the risk-neutral yield dynamics are given by the solution to the following stochastic differential equation (SDE):

$$dy_t(\tau) = b_t(\tau)dt + \omega_t(\tau)dZ_t, \qquad t \geq 0, \tag{37}$$

where Z is a $\mathbb{Q}$ standard Brownian motion. We refer to $b(\tau)$ as the risk-neutral drift process for the $\tau-$yield, while we refer to $\omega_t(\tau)$ as the $\tau-$yield's volatility process. Importantly, we do not need to specify the $\mathbb{Q}$ dynamics of these processes when our only goal is to produce an entire arbitrage-free yield curve from a few given market quotes.

Let $b_t(T)$ be the market price of a bond. By the definition of yield to maturity $y_t(\tau)$:

$$b_t(T) = B(y_t(\tau), \tau), \qquad t \geq 0, \tau \geq 0, \tag{38}$$

where the bond pricing function is defined as:

$$B(y, \tau) = e^{-y\tau}, \qquad y \in \mathbb{R}, \tau \geq 0. \tag{39}$$

The absence of arbitrage implies that at each time t, each bond's price grows in expectation at the short rate r_t. As a result, we obtain the following no arbitrage constraint on yields:

$$\frac{\partial}{\partial \tau} B(y_t(\tau), \tau) = \frac{\omega_t^2(\tau)}{2} \frac{\partial^2}{\partial y^2} B(y_t(\tau), \tau) + b_t(\tau) \frac{\partial}{\partial y} B(y_t(\tau), \tau) - r_t B(y_t(\tau), \tau), \qquad \tau \geq 0. \tag{40}$$

From (39), we have the following three greeks:

1. $\frac{\partial}{\partial \tau} B(y_t(\tau), \tau) = -y_t(\tau) B(y_t(\tau), \tau)$
2. $\frac{\partial}{\partial y} B(y_t(\tau), \tau) = -\tau B(y_t(\tau), \tau)$
3. $\frac{\partial^2}{\partial y^2} B(y_t(\tau), \tau) = \tau^2 B(y_t(\tau), \tau)$.

Substituting the 3 greeks in (40) and dividing out $-B(y_t(\tau), \tau)$ implies:

$$y_t(\tau) = r_t + b_t(\tau)\tau - \frac{\omega_t^2(\tau)}{2} \tau^2, \qquad \tau \geq 0. \tag{41}$$

A specification of the risk-neutral drift and diffusion processes governing yields determines an arbitrage-free yield curve. For example, suppose that the risk-neutral drift and diffusion processes are both independent of τ, i.e.

$$b_t(\tau) = b_t, \qquad \omega_t(\tau) = \omega_t, \qquad t \geq 0, \tau \geq 0. \tag{42}$$

Then the SDE (37) implies that the yield curve moves continuously and only by parallel shifts. Substituting (42) in (41) implies that the resulting yield curve is quadratic in τ opening down[5] .

$$y_t(\tau) = r_t + b_t \tau - \frac{\omega_t^2}{2} \tau^2, \qquad \tau \geq 0. \tag{43}$$

Given the market yields of three bonds at time t, the numerical values of the processes r_t, b_t and ω_t can be determined. Note that the variation over time of b and ω^2 is entirely consistent with the model. This consistency is in stark contrast to parameter variation in short rate models. Unpredictable parameter variation over time in short rate models requires an alternative dynamical specification, which will in general lead to a different functional form of the yield curve. While market models enjoy this advantage for the problem of yield curve construction, they can only be used to value bonds. In contrast, a short rate model can be used to value bonds and other derivatives consistently.

If more than three yields are observed, then one can either do a least squares fit of (43) or re-parametrize (41 appropriately. There are various specifications of the dependence of the processes b_t and ω_t on τ which either lead to more realistic behavior of yields eg. mean reversion and/or lead to closed form solutions for the yield curve.

[5] It can be shown more generally that when a yield curve moves only by parallel shifts, then whether its risk-neutral dynamics are continuous or not, the absence of arbitrage forces the yield curve to be concave.

3.2 Market Model for Implied Volatilities

In common with the last subsection, we assume that the short interest rate is stochastic. However, our focus is on pricing swaptions, not bonds. At some given term and tenor, we assume that the market gives us implied normal volatilities of swaptions at a finite number of strike rates. The objective is to connect the dots so as to produce a full volatility smile.

We assume no arbitrage and that $\mathbb{P}$ is the real world probability measure. Let $\mathbb{Q}^a$ be the martingale measure equivalent to $\mathbb{P}$, which arises when the forward starting annuity is taken to be the numeraire.

Suppose that under $\mathbb{Q}^a$, the forward swap rate process F solves the following stochastic differential equation (SDE):

$$dF_t = \sqrt{V_t}dW_t, \qquad t \geq 0, \tag{44}$$

where W is a $\mathbb{Q}^a$ standard Brownian motion. The stochastic process V is the instantaneous normal variance rate of F. In contrast to classical stochastic volatility models such as SABR, we do not directly specify the dynamics of this process. Let $\eta_t(m)$ be the normal implied volatility at the same fixed term and tenor as the forward starting annuity. The argument m is called moneyness and is given by the fraction $\frac{F_t-K}{\eta_t(m)}$. Since η^2 is the annualized variance rate, $\eta^2\tau$ is the non-annualized variance of the terminal forward swap rate. Its square root $\eta\sqrt{\tau}$ is the standard deviation of the terminal forward swap rate. If we drop the $\sqrt{\tau}$, then η is the annualized standard deviation of the terminal forward swap rate, i.e. what this standard deviation would be if the term were one year. Hence, the moneyness measure m is the number of annualized standard deviations $\eta_t(m)$ that the underlying forward rate F exceeds the strike rate K, as indicated previously.

To compensate for the absence of a specification of the instantaneous normal variance rate V, we suppose that under $\mathbb{Q}^a$, the implied volatility process $\eta_t(m)$ is the solution to the following stochastic differential equation (SDE)

$$d\eta_t(m) = \mu_t(m)dt + \omega_t(m)dZ_t, \qquad m \in \mathbb{R}, t \geq 0, \tag{45}$$

where Z is a $\mathbb{Q}^a$ standard Brownian motion. We refer to $\mu_t(m)$ as the FSA measure drift process and we refer to $\omega_t(m)$ as the volvol process. Let $\rho_t \in [-1, 1]$ be the bounded stochastic process governing the correlation between the two standard Brownian motions W and Z at time t. The processes F and ρ are scalar-valued stochastic processes, while the processes $\eta_t(m)$, $\mu_t(m)$ and $\omega_t(m)$ are function-valued stochastic processes.

Recall that the swaption's value depends on F_t and K only though the excess $X_t = F_t - K$. Subtracting K from F in (44) implies:

$$dX_t = \sqrt{V_t}dW_t, \qquad t \geq 0. \tag{46}$$

The absence of arbitrage implies that at each time t, each swaption's price is a local martingale. As a result, we obtain the following no arbitrage constraint on implied volatilities $\eta_t(m)$:

$$\frac{\partial}{\partial \tau}C(\eta_t(m), X_t, \tau) = \left[\frac{\omega_t^2(m)}{2}\frac{\partial^2}{\partial \eta^2} + \rho_t\omega_t(m)\sqrt{V_t}\frac{\partial^2}{\partial \eta \partial x} + \frac{V_t}{2}\frac{\partial^2}{\partial x^2} + u_t(m)\frac{\partial}{\partial \eta}\right] C(\eta_t(m), X_t, \tau), \tag{47}$$

for $m \in \mathbb{R}$.

For the Bachelier call formula, vega and gamma are related by:

$$\frac{\partial}{\partial \eta} C(\eta_t(m), x, \tau) = \eta_t(m)\tau\Gamma(\eta_t(m), x, \tau), \qquad m \in \mathbb{R}, x \in \mathbb{R}, \tau > 0. \tag{48}$$

Substituting (48) in (49) implies that:

$$\frac{\partial}{\partial \tau} C(\eta_t(m), X_t, \tau) = \left[\frac{\omega_t^2(m)}{2} \frac{\partial^2}{\partial \eta^2} + \rho_t \omega_t(m) \sqrt{V_t} \frac{\partial^2}{\partial \eta \partial x} + \frac{1}{2} \left[V_t + 2\eta_t(m)\mu_t(m)\tau \right] \frac{\partial^2}{\partial x^2} \right] C(\eta_t(m), X_t, \tau), \tag{49}$$

for $m \in \mathbb{R}$. The coefficient of $\frac{\partial^2}{\partial x^2} C(\eta_t(m), X_t, \tau)$ is $V_t + 2\eta_t(m)\mu_t(m)\tau$. Notice that V_t is the short term ATM implied variance, $2\eta_t(m)$ is the derivative of the variance rate $\eta_t^2(m)$ w.r.t. the normal volatility $\eta_t(m)$, and $\mu_t(m)\tau$ is the expected change in the volatility over a period of length τ. Hence one can interpret $V_t + 2\eta_t(0)\mu_t(0)\tau$ as the ATM implied variance rate at term τ. We will see it plays the same role as the short rate in a yield curve construction.

The normal implied variance rate $\eta^2(m)$ at moneyness $m \in \mathbb{R}$ has the property that it balances the maturity derivative[6] with the gamma trading profits, i.e.:

$$\frac{\partial}{\partial \tau} C(\eta_t(m), x, \tau) = \frac{\eta_t^2(m)}{2} \Gamma(\eta_t(m), x, \tau), \qquad m \in \mathbb{R}, x \in \mathbb{R}, \tau > 0. \tag{50}$$

The appendix proves that for any function $f : \mathbb{R} \mapsto \mathbb{R}$ and for $n = 0, 1, \ldots$:

$$(D_\eta D_x^{-1})^n \frac{f\left(\frac{x}{\eta}\right)}{\eta} = \left(-\frac{x}{\eta}\right)^n \frac{f\left(\frac{x}{\eta}\right)}{\eta}, \qquad \eta > 0, x \in \mathbb{R}. \tag{51}$$

Hence when $f(z) = \frac{N'(z/\sqrt{\tau})}{\sqrt{\tau}}$:

$$\frac{f\left(\frac{x}{\eta}\right)}{\eta} = \frac{N'\left(\frac{x}{\eta\sqrt{\tau}}\right)}{\eta\sqrt{\tau}} = \Gamma(\eta, x, \tau). \tag{52}$$

As a result:

$$(D_\eta D_x^{-1})^n \Gamma(\eta, x, \tau) = (-m)^n \Gamma(\eta, x, \tau), \qquad \eta > 0, x \in \mathbb{R}, \tau > 0, \tag{53}$$

where recall that $m \equiv x/\sqrt{\eta}$. This is analogous to the obvious statement that for $n = 0, 1, \ldots$:

$$D_y^n B(y, \tau) = (-\tau)^n B(y, \tau), \qquad y \in \mathbb{R}, \tau \geq 0, \tag{54}$$

where $B(y, \tau) = e^{-y\tau}$. Equations (53) and (54) are the basis of our statement that the function Γ relating swaption gamma to η and x at each $\tau > 0$ has the same role in generating the volatility smile, η^2 vs. $m = x/\eta$, as the function B relating bond price to y and τ has in generating a yield curve, y vs. τ.

[6] If the dependence of η on τ were made explicit, then the LHS of (50) would hold constant the $\tau's$ inside $\eta(x/\sqrt{\tau}, \tau)$.

Evaluating (53) at $n = 1$ and $n = 2$ leads to the following two greeks:

$$\frac{\partial^2}{\partial\eta\partial x}C(\eta, x, \tau) = D_\eta D_x^{-1}\Gamma(\eta, x, \tau) = -m\Gamma(\eta, x, \tau). \tag{55}$$

$$\frac{\partial^2}{\partial\eta^2}C(\eta, x, \tau) = (D_\eta D_x^{-1})^2\Gamma(\eta, x, \tau) = m^2\Gamma(\eta, x, \tau). \tag{56}$$

Substituting the three greek relations (50), (55), and (56) in the no arbitrage condition (49) and dividing out $\Gamma(\eta, x, \tau)/2$ implies:

$$\eta_t^2(m) = [V_t + 2\eta_t(m)\mu_t(m)\tau] - \rho_t\omega_t(m)\sqrt{V_t}m + \frac{\omega_t^2(m)}{2}m^2, \qquad m \in \mathbb{R}. \tag{57}$$

A specification of the drift process $\mu_t(m)$ and of the volvol process $\omega_t(m)$ governing normal implied volatilities determines an arbitrage-free implied variance rate curve. For example, suppose that the drift and vol-vol processes are both independent of m, i.e.

$$\mu_t(m) = \mu_t, \qquad \omega_t(m) = \omega_t, \qquad t \geq 0, m \in \mathbb{R}. \tag{58}$$

Then from (45), the normal implied volatility curve $\eta(m)$ moves continuously and only by parallel shifts[7]. Suppose furthermore that $\mu_t = 0$. Substituting $\mu_t = 0$ and (58) in (57) implies that the resulting implied variance rate curve is quadratic in m opening up:

$$\eta_t^2(m) = V_t - \rho_t\omega_t\sqrt{V_t}m + \frac{\omega_t^2}{2}m^2, \qquad m \in \mathbb{R}. \tag{59}$$

If $\mu_t(m)$ is not zero but is independent of m as in (58), then (59) becomes:

$$\eta_t^2(m) = [V_t + 2\eta_t(m)\mu_t\tau] - \rho_t\omega_t\sqrt{V_t}m + \frac{\omega_t^2}{2}m^2, \qquad m \in \mathbb{R}. \tag{60}$$

It is straightforward to use the quadratic root formula in (60) to determine the dependence of $\eta_t(m)$ on m at term τ. In theory, the market quote of a short term ATM implied vol determines V_t. As in the case of yields, the market quotes of three co-terminal implied volatilities can be used to determine the numerical values of μ_t, ρ_t, and ω_t.

3.3 Comparing Market Models

It is interesting to compare the arbitrage-free yield curve that arises when all yields are driven by a continuous single factor:

$$y_t(\tau) = r_t + b_t(\tau)\tau - \frac{\omega_t^2(\tau)}{2}\tau^2, \qquad \tau \geq 0, \tag{61}$$

[7]Note that by Itô's formula, the volatility smile $\eta^2(m)$ will not move by parallel shifts. Also it can be shown more generally that when an implied volatility curve $\eta(m)\sqrt{\tau}$ is flat and moves only by unpredictable parallel shifts, then whether its risk-neutral dynamics are continuous or not, there is arbitrage.

with the arbitrage-free volatility smile that arises when all implied volatilities are driven by a continuous single factor:

$$\eta_t^2(m) = [V_t + 2\eta_t(m)\mu_t(m)\tau] - \rho_t\omega_t(m)\sqrt{V_t}m + \frac{\omega_t^2(m)}{2}m^2, \qquad m \in \mathbb{R}. \tag{62}$$

Both curves have three components. Setting $\tau = 0$ in (61) gives the short rate, which is the first component in (61). Analogously, setting $m = 0$ in (62) gives $V_t + 2\eta_t(0)\mu_t(0)\tau$ which is the ATM implied variance rate at term τ. The last component in both expressions is due to stochastic variation in the yield or the volatility. The middle component of the yield curve is due to the drift of yields, while the middle component of the volatility smile is due to the quadratic covariation of implied volatility and the forward swap rate. If we change measure away from $\mathbb{Q}$ to a measure under which yields are driftless, then the drift in yields under $\mathbb{Q}$ would all be due to the covariation of yields with the Radon Nikodym derivative. As a result, the middle component in both curves can be seen as due to quadratic covariation.

It is also interesting to compare the quadratic yield curve in the last section:

$$y_t(\tau) = r_t + b_t\tau - \frac{\omega_t^2}{2}\tau^2, \qquad \tau \geq 0, \tag{63}$$

with the quadratic volatility smile in (59):

$$\eta_t^2(m) = V_t - \rho_t\omega_t\sqrt{V_t}m + \frac{\omega_t^2}{2}m^2, \qquad m \in \mathbb{R}. \tag{64}$$

The reason that the yield curve opens down while the volatility smile opens up is due to the fact that the relation between bond prices and yields is opposite in sign to the relation between swaption prices and implied variances.

4 Summary and Extensions

In this paper, we found particular measures of implied volatility and moneyness of a European swaption, so that the resulting volatility smile is analogous to the yield curve for zero coupon bonds. The relation was used to develop arbitrage-free curves in both cases. There are various avenues for future research. One can see if the long maturity behavior of yields discovered by Dybvig, Ingersoll, and Ross[7] has its counterpart for normal implied volatilities at extreme strikes. One can try to see if the analogous curve constructions are available when yields and implied volatilities can jump. One can also explore the similarity of the arbitrage if flat curves move only by unpredictable parallel shifts. One can redo the analysis with Black implied volatilities or even implied volatilities from other models. One can bring in other instruments and associated market rates such as coupon bonds and yields or corporate bonds and credit spreads. Finally, one can explore the role of fundamental solutions in the analogy. In the interests of brevity, these extensions are best left for future research.

Appendix

In this appendix, we provide a short proof that for any sufficiently differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ and for $n = 0, 1, \ldots$:

$$(D_s D_x^{-1})^n \frac{f\left(\frac{x}{s}\right)}{s} = \left(-\frac{x}{s}\right)^n \frac{f\left(\frac{x}{s}\right)}{s}, \qquad s > 0, x \in \mathbb{R}. \tag{65}$$

We first show the result holds for $n = 1$, i.e.

$$D_s D_x^{-1} \frac{f\left(\frac{x}{s}\right)}{s} = \left(-\frac{x}{s}\right) \frac{f\left(\frac{x}{s}\right)}{s}, \qquad s > 0, x \in \mathbb{R}. \tag{66}$$

The LHS is:

$$D_s D_x^{-1} \frac{f\left(\frac{x}{s}\right)}{s} = D_s \int\limits_{-\infty}^{x} \frac{f\left(\frac{y}{s}\right)}{s} dy = D_s \int\limits_{-\infty}^{\frac{x}{s}} f(z) dz = \frac{-\frac{x}{s} f\left(\frac{x}{s}\right)}{s}, \tag{67}$$

by the fundamental theorem of calculus and the chain rule. Thus, the result (66) does hold for any function f of $\frac{x}{s}$, when the function f is divided by the scale factor $s > 0$. Notice from (67) that the effect of applying the operator $D_s D_x^{-1}$ to the fraction $\frac{f}{s}$, where the numerator f just depends on $\frac{x}{s}$ is another fraction $\frac{g}{s}$, where the numerator $g(z) \equiv -z f(z)$ just depends on $z = \frac{x}{s}$. As a result, one can apply the operator $D_s D_x^{-1}$ to the fraction $\frac{g}{s}$ to obtain:

$$\left(D_s D_x^{-1}\right)^2 \frac{f\left(\frac{x}{s}\right)}{s} = \frac{-\frac{x}{s} g\left(\frac{x}{s}\right)}{s} = \frac{\left(-\frac{x}{s}\right)^2 f\left(\frac{x}{s}\right)}{s}. \tag{68}$$

Repeating this exercise $n - 2$ times leads to the desired result (65). Re-arranging (65) implies that for any sufficiently differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ and for $n = 0, 1, \ldots$:

$$s^n D_s^n D_x^{-n} \frac{f\left(\frac{x}{s}\right)}{s} = (-x)^n \frac{f\left(\frac{x}{s}\right)}{s}, \qquad s > 0, x \in \mathbb{R}. \qquad \mathbf{Q.E.D.} \tag{69}$$

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