Multiset Poker

Daniel Conus and Garth Isaak Lehigh University

MAA-EPADEL - November 12, 2016

4 Cards: Q_1, Q_2, K_1, K_2 - Pick 2

Set	Prob
$Q_1 Q_2$	$\frac{1}{6}$
$Q_1 K_1$	$\frac{1}{6}$
$Q_1 K_2$	$\frac{1}{6}$
$Q_2 K_1$	$\frac{1}{6}$
$Q_2 K_2$	$\frac{1}{6}$
$K_1 K_2$	$\frac{1}{6}$

Multi Set	Prob
$Q_1 Q_2$	$\frac{1}{10}$
$Q_1 K_1$	$\frac{1}{10}$
$Q_1 K_2$	$\frac{1}{10}$
$Q_2 K_1$	$\frac{1}{10}$
$Q_2 K_2$	$\frac{1}{10}$
$K_1 K_2$	$\frac{1}{10}$
$Q_1 Q_1$	$\frac{1}{10}$
$Q_2 Q_2$	$\frac{1}{10}$
$K_1 K_1$	$\frac{1}{10}$
$K_2 K_2$	$\frac{1}{10}$

4 Cards: Q_1, Q_2, K_1, K_2 - Multiset

Multi Set	Prob	Replace ment	Prob	Two Decks	Prob
$Q_1 Q_2$	$\frac{1}{10}$	$Q_1 Q_2$	$\frac{2}{16}$	$Q_1 Q_2$	$\frac{4}{28}$
$Q_1 K_1$	$\frac{1}{10}$	$Q_1 K_1$	$\frac{2}{16}$	$Q_1 K_1$	$\frac{4}{28}$
$Q_1 K_2$	$\frac{1}{10}$	$Q_1 K_2$	$\frac{2}{16}$	$Q_1 K_2$	$\frac{4}{28}$
$Q_2 K_1$	$\frac{1}{10}$	$Q_2 K_1$	$\frac{2}{16}$	$Q_2 K_1$	$\frac{4}{28}$
$Q_2 K_2$	$\frac{1}{10}$	$Q_2 K_2$	$\frac{2}{16}$	$Q_2 K_2$	$\frac{4}{28}$
$K_1 K_2$	$\frac{1}{10}$	$K_1 K_2$	$\frac{2}{16}$	$K_1 K_2$	$\frac{4}{28}$
$Q_1 Q_1$	$\frac{1}{10}$	$Q_1 Q_1$	$\frac{1}{16}$	$Q_1 Q_1$	$\frac{1}{28}$
$Q_2 Q_2$	$\frac{1}{10}$	$Q_2 Q_2$	$\frac{1}{16}$	$Q_2 Q_2$	$\frac{1}{28}$
$K_1 K_1$	$\frac{1}{10}$	$K_1 K_1$	$\frac{1}{16}$	$K_1 K_1$	$\frac{1}{28}$
$K_2 K_2$	$\frac{1}{10}$	$K_2 K_2$	$\frac{1}{16}$	$K_2 K_2$	$\frac{1}{28}$

Use 5 Cards: C, Q_1, Q_2, K_1, K_2 - Pick 2

Set	Multi Set	Prob
$Q_1\,Q_2$	$Q_1\,Q_2$	$\frac{1}{10}$
$Q_1\,K_1$	$Q_1\,K_1$	$\frac{1}{10}$
$Q_1\,K_2$	$Q_1\,K_2$	$\frac{1}{10}$
$Q_1\,K_2$	$Q_2\,K_1$	$\frac{1}{10}$
$Q_2\,K_2$	$Q_2\,K_2$	$\frac{1}{10}$
$K_1\,K_2$	$K_1\,K_2$	$\frac{1}{10}$
$C\;Q_1$	$Q_1\,Q_1$	$\frac{1}{10}$
$C\;Q_2$	$Q_2\,Q_2$	$\frac{1}{10}$
$C\;K_1$	$K_1\,K_1$	$\frac{1}{10}$
$C\;K_2$	$K_2\,K_2$	$\frac{1}{10}$

Notation

$\binom{n}{k}$ = # *k elt subsets from n set:* $\binom{52}{5} = \frac{52!}{5!47!} = 2,598,560$

$\left(\!\binom{n}{k}\!\right)$ = # *k elt multisets from n set:* $\left(\!\binom{52}{5}\!\right) = \frac{(52+4)!}{52!4!} = 3,819,816$

$\binom{n}{k_1,k_2,\dots,k_t}$ = *multinomial coefficient:* $\binom{13}{1,2,10} = \frac{13!}{1!2!10!} = 858$

Fact

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Standard Bijection:

Map 4 element sets from $\{1,2,3,4,5,6,7,8,9\}$ *to*
4 element multisets from $\{1,2,3,4,5,6\}$ *(and reverse)*

set	stars and bars	multiset
1, 3, 4, 6	$* \vert * * \vert * \vert \vert \vert$	1, 2, 2, 3
1, 4, 6, 9	$* \vert \vert * \vert * \vert \vert *$	1, 3, 4, 6
2, 3, 4, 8	$\vert * * * \vert \vert \vert * \vert$	2, 2, 2, 5

Linear order of augmented deck
Low to high ranks: $2, 3, \ldots, 9, 10, J, Q, K, A, C$
Low to high within ranks: $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$

The standard bijection

Dealt hand		Multiset hand
$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$	$\Longleftrightarrow$	$3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit$
$3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit C\clubsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit 10\spadesuit 10\spadesuit A\clubsuit$
$3\diamondsuit 5\clubsuit J\heartsuit C\diamondsuit C\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit J\clubsuit A\heartsuit A\heartsuit$

Knight's Bijection:

Map 4 element sets from $\{1,2,3,4,5,6,7,8,9\}$ *to*
4 element multisets from $\{1,2,3,4,5,6\}$ *(and reverse)*

Set from $\{1,2,3,4,5,6,7,8,9\}$ *avoiding* $\{7,8,9\}$ *maps to itself*

set	multiset
1, 3, 4, 6	1, 3, 4, 6

Knight's Bijection:

Map 4 element sets from $\{1,2,3,4,5,6,7,8,9\}$ *to 4 element multisets from* $\{1,2,3,4,5\}$ *(and reverse)*

- Place $\{7,8,9\}$ in positions 1,2,3
- Fill in other elements in open spaces (in order)
- $\{7,8,9\}$ assume value of first element to their right

set	rearrange	multiset
$2,4,5,8$	$2,8,4,5$	$2,4,4,5$
$2,4,7,8$	$7,8,2,4$	$2,2,2,4$

The Knight's bijection

- A hand with no knight maps to itself
- Place knights in their location $C\clubsuit$, $C\diamondsuit$, $C\heartsuit$, $C\spadesuit$ $=$ 1,2,3,4 left to right
- Place remaining cards in open spaces in order
- Knights take value of first regular card to their right

$$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit \iff 3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$$

$$C\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \iff 3\diamondsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$3\diamondsuit C\diamondsuit C\heartsuit 5\clubsuit J\heartsuit \iff 3\diamondsuit 5\clubsuit 5\clubsuit 5\clubsuit J\heartsuit$$

Knight's bijection C : $\binom{[n+k-1]}{k} \Leftrightarrow \left(\!\!\binom{[n]}{k}\!\!\right)$

- Any set avoiding knights maps to itself
- Place knights in their location
- Place regular elements in order in open spots
- Knights take value of first regular element to their right

Proof idea:

- Stars and bars bijection with 'extra' elements as stars and 'regular' elements as bars
- Multiset version of Vandermonde identity
  $\binom{n+k-1}{k} = \sum \binom{n}{i}\binom{k-1}{k-i}$ $\qquad$ $\left(\!\!\binom{n}{k}\!\!\right) = \sum \binom{n}{i} \left(\!\!\binom{i}{k-i}\!\!\right)$

Playing poker with Knight's bijection

- No 'numerical' computations needed
- 'Normal' hands are themselves
- No 2 players can get the same card
- At most 4 instances of duplicated cards
-
- High card 'beats' one pair

General Poker Games

3 'Deals'

- *Multiple Decks (t decks)*
- *Multiset bijection*
- *Dealing with replacement*

- r ranks
- s suits
- hand size h

limit as $t \to \infty$ multideck is dealing with replacement

Notation for general poker

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle \qquad r = \sum p_i$ and $h = \sum i \cdot p_i$

Regular 13 rank poker:

$\langle 0^{10}, 1^2, 2^0, 3^1 \rangle$ is 3 of a kind

$\langle 0^{10}, 1^1, 2^2 \rangle$ is 2 pair

With $r = 17$ ranks and hand size $h = 13$:

$\langle 0^{10}, 1^4, 2^0, 3^3 \rangle$ is three 3 of a kinds

$\langle 0^{11}, 1^3, 2^1, 3^0, 4^2 \rangle$ is two 4 of a kind and a pair

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

t decks

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \binom{st}{i}^{p_i}}{\binom{rst}{h}}$$

Multiset

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \left(\!\!\binom{s}{i}\!\!\right)^{p_i}}{\left(\!\!\binom{rs}{h}\!\!\right)}$$

Dealing with replacement

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \binom{h}{(0)^{p_0}, (1)^{p_1}, (2)^{p_2}, (3)^{p_3}, \ldots} \cdot s^h}{(rs)^h}$$

13 ranks, 4 suits, 5 card hands

2 pair (including flushes): $\langle 0^{10}, 1^1, 2^2 \rangle$

5 decks:

$$\frac{\binom{13}{10,1,2} \cdot \binom{20}{1}\binom{20}{2}^2}{\binom{260}{5}}$$

multiset:

$$\frac{\binom{13}{10,1,2} \cdot \left(\!\binom{4}{1}\!\right)\left(\!\binom{4}{2}\!\right)^2}{\left(\!\binom{52}{5}\!\right)}$$

Dealing with replacement:

$$\frac{\binom{13}{10,1,2} \cdot \binom{5}{1,2,2} \cdot 4^5}{52^5}$$

17 ranks, 6 suits, 13 card hands
two 4 of a kind and a pair (including flushes): $\langle 0^{11}, 1^3, 2^1, 3^0, 4^2 \rangle$

1 deck:

$$\frac{\binom{17}{11,3,1,2} \cdot \binom{6}{1}^3 \binom{6}{2} \binom{6}{4}^2}{\binom{102}{13}}$$

multiset:

$$\frac{\binom{17}{11,3,1,2} \cdot \left(\!\!\binom{6}{1}\!\!\right)^3 \left(\!\!\binom{6}{2}\!\!\right) \left(\!\!\binom{6}{4}\!\!\right)^2}{\left(\!\!\binom{102}{13}\!\!\right)}$$

Dealing with replacement:

$$\frac{\binom{17}{11,3,1,2} \cdot \binom{13}{1,1,1,2,4,4} \cdot 6^{13}}{102^{13}}$$

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

t decks

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \binom{st}{i}^{p_i}}{\binom{rst}{h}}$$

Multiset

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \left(\!\binom{s}{i}\!\right)^{p_i}}{\left(\!\binom{rs}{h}\!\right)}$$

Dealing with replacement

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \binom{h}{(0)^{p_0}\ (1)^{p_1}\ (2)^{p_2}\ (3)^{p_3} \ldots} \cdot s^h}{(rs)^h}$$

Multiset vs. regular probabilities (as percents %)

	multiset	regular
Straight flush	.001	.001
5 kind flush	.001	0
4 kind flush	.016	0
full house flush	.016	0
5 kind	.02	0
3 kind flush	.09	0
2 pair flush	.09	0
flush	.13	.20
straight	.27	.39
pair flush	.30	0
4 kind	.56	.02
full house	.80	.14
3 kind	7.10	2.87
2 pair	8.90	4.75
High card	34.10	49.68
1 pair	47.62	42.3

regular poker hands $\binom{52}{5} = 2,598,960$

multiset poker hands $\left(\!\binom{52}{5}\!\right) = \binom{56}{5} = 3,819,816$

- A hand with no knight maps to itself
- Place knights in their location $C\clubsuit$, $C\diamondsuit$, $C\heartsuit$, $C\spadesuit$ $=$ 1,2,3,4 left to right
- Place remaining cards in open spaces in order
- Knights take value of first regular card to their right

The Knight's bijection

$$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit \iff 3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$$

$$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \iff 3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$C\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \iff 3\diamondsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$3\diamondsuit C\diamondsuit 5\clubsuit C\spadesuit J\heartsuit \iff 3\diamondsuit 5\clubsuit 5\clubsuit J\heartsuit J\heartsuit$$

$$3\diamondsuit C\diamondsuit C\heartsuit 5\clubsuit J\heartsuit \iff 3\diamondsuit 5\clubsuit 5\clubsuit 5\clubsuit J\heartsuit$$