Math Seminar on Godel's Thm. Summer 1984 — by Garth Isaak

Kurt Gödel's Incompleteness Theorems first presented in his 1931 paper "On Formally Undecidable Propositions in <u>Principia Mathematica</u> and Related Systems I" [1] represent an important turning point in the study of foundations of mathematics. It states that to every consistent axiomatic formulation of number theory there are undecidable propositions. More formally it says:

> To every $\omega$-consistent recursive class $\kappa$ of formulae there correspond recursive class-signs r, such that neither v Gen r nor Neg(v Gen r) belongs to Flg($\kappa$) (where v is a free variable of r).2

This theorem along with its corollary that the consistency of the formal system cannot be proven from within the system put an end to David Hilbert's program of Formalism; the attempt to codify all of mathematics based on a finite number of simple axioms and some basic rules of logical inference. Gödel's Theorems have been the basis for many discussions since ranging from proof theory to artificial intelligence.

Expositions of Gödel's proof tend to be either a simple analogy to the Epimenides paradox or a full bore detailed explanation of the details of the proof. These detailed descriptions, even when they attempt to shed technical jargon tend to be long and difficult to comprehend. Conversely, the Epimenides paradox can be simply stated in a modern version as 'This sentence is false.' This reveals some of the trick to Gödel's proof, but does not reveal the full beauty of his creation.

In this paper, an allegory is presented which attempts to bridge the gap between extreme simplicity and confusing detail. By providing an overview of the general structure, this allegory can be a stepping stone toward understanding the intricacies of Gödel's proof. The story requires acceptance of some strange concepts and of a computer performing some black box tricks; these are included to simplify matters and expose the skeleton of the proof. Without further explanation we shall conjure up some mental images.

Imagine, if you will, that sometime in the future humankind perfects space travel and ventures out to discover a new race of beings. Fortunately these beings breathe the same air as we do and are suited for the same types of climate, gravity, etc. However, one problem remains before the two races can work together, communication.

You see, it seems that this new race, whom we will call Godelians, communicate in quite a strange way. They build little structures out of a set of blocks, which curiously enough resemble our familiar Lego brand blocks. Each Godelian carries around a small bag of blocks and when they wish to communicate, builds a structure which presents a thought. In the same manner we would write a sentence to present a thought. Now, each Godelian structure carries a unique meaning, unlike our language.

The Godelians use five basic blocks in their structures and these pieces can only be put together following several obvious rules of construction. Only, if we may coin a word, 'constructable' structures are meaningful to the Godelians even though designs for others can be drawn. The whole of Godelian thought is represented by constructable structures. This does not limit thought as new, constructable structures are continually being discovered.

In order to translate between these two quite different methods of communication a computer is programed which takes an English sequence of letters, spaces, and punctuation marks and creates a diagram, or blueprint of a Lego structure. Each such blueprint uniquely represents this 'phrase'. We won't detail how this works, rather will just marvel at the ingenuity of this magical black box. One of the translators, call him Hilbert, wants to know just how good a job this computer does. He wishes to show that every meaningful human idea is translated into a parallel meaningful Godelian idea.

Hilbert believes that any blueprint of a meaningful human idea can be checked simply by playing with the blocks in a routine fashion until an answer is found; either a design can be built or it cannot and therefore any particular human idea is meaningful to the Godelians or it is not. It is of course also hoped that meaningless human phrases are translated into meaningless structures.

Unfortunately, the routine process of determining which structures can be built with Godelian Lego blocks becomes tedious, and despite attempts at simplification the task seems formidable. Very quickly human ideas arise for which no one can tell if they are construcable. Godelians themselves can spend years trying to decide whether or not a particular blueprint can be constructed. As work continues to discover the constructability of these blueprints, a young aid of Hilbet's named Kurt discovers a startling fact: There are meaningfull human ideas which cannot be expressed to the Godelians using Lego structures.

In order to follow Kurt's reasoning, we must first realize that words only represent ideas, they are not the ideas themselves. Similarly Lego structures represent ideas and thoughts of the Godelians. It is important to keep this distinction between words and ideas clear. In the demonstration of his statement, Kurt linked an idea and the words representing it in a clever manner. He wrote the sentence 'The blueprint representing this sentence is not constructable by the Godelians.' In this one step Kurt had accomplished the task of demonstrating that the perfect translating machine could not be built. For if the Godelians could construct the blueprint representing this idea, they would be faced with the contradictory notion that they can and cannot construct the same structure. Thus we assume that the structure representing this sentence is not constructable, which leads to the conclusion that the translating computer cannot do a complete job. The incompleteness lies in the fact that the meaningful English sentence 'The blueprint representing this sentence is not constructable by the Godelians' cannot be translated into a Lego structure by the computer. Thus there can be no perfect translation between the thoughts of the two races.

Having briefly examined a story which appears to have nothing in common with mathematics, let alone Gödel's Theorem, let us examine the allegory. The key point of the allegory is to present the ingenious paradoxical structure Gödel created by the clever use of a numbering scheme and a diagonal arguement.

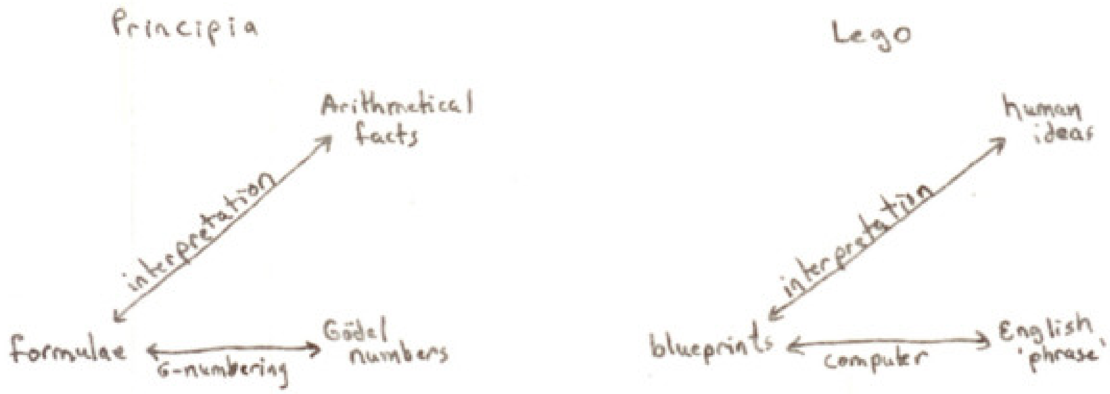
Gödel's Theorem talks about propositions in Principia Mathematica and other formal systems. The formal systems referred to consist of a finite collection of symbols and rules of formation and transformation for manipulating these symbols along with a set of primative basic formulae called axioms.[3] A theorem of such a system is arrived at by starting with axioms and applying the transformation rules a finite number of times. The system itself consists only of these meaningless strings of symbols. In our story, the five basic Lego type blocks represent the basic statements or axioms, and the process of constructing structures represent the rules of transformation. Thus we see that a theorem in a formal system is likened to a constructable structure in the Godelian world.

The similarity between these systems continues in the fact that although appearing to be meaningless forms, each system has an interpretation. Typically formal systems such as the one created by Bertrand Russell and Alfred North Whitehead in Principia Mathematica describe number theory. In this interpretation symbols such as '=' have the common interpretation 'is equal to', and theorems in Principia represent common numerical truths such as '1+1=2'. Similarly, the structures of Lego type building blocks can be interpreted as representing a human thought or idea. We can diagram this relationship as follows:


We note that in Principia and its relation to arithmetic, a statement in number theory is true if the formula representing it in Principia is demonstrable. That is if the formula can be derived from the axioms and rules of transformation. Similarly a sentence is meaningful in English when the Godelian structure representing it is constructable.

The next comparison represents the genius of Gödel. In our story we have attributed some rather amazing capabilities to the translation program of the computer. It is able to create a unique mapping which links each conceivable structure to a sequence of letters, spaces, and punctuation marks. It is a similar map in Gödel's proof that provides the self scrutiny by a formal system of itself which is the crux of the proof.[4]

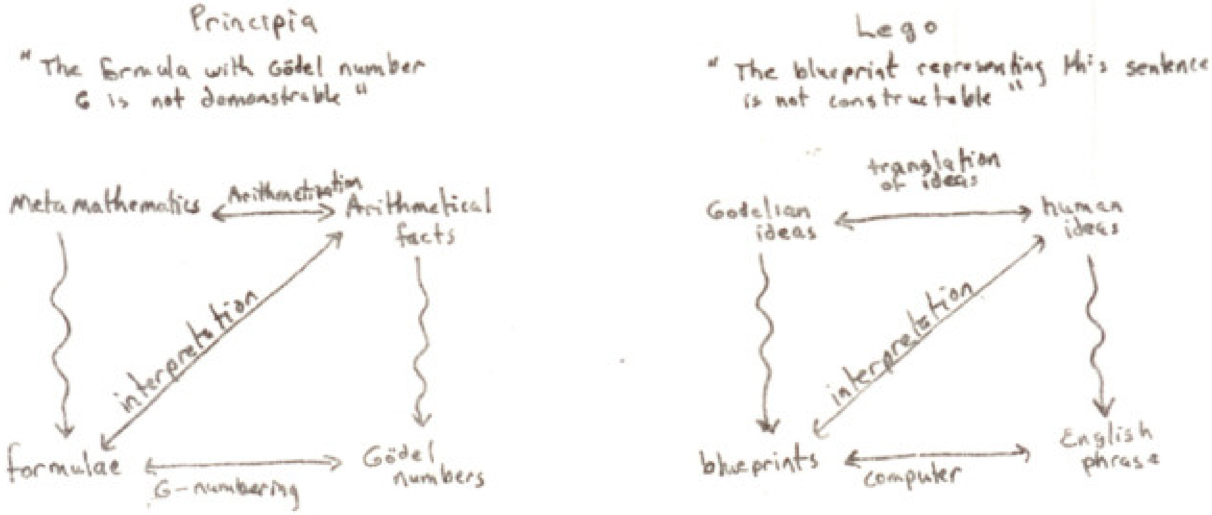
In this process called Gödel numbering each symbol in a formal system is assigned a unique prime natural number. This Gödel numbering along with the interpretation of a formal system as representative of number theory forms a self referential loop, which in turn allows the Epimenides paradox to create a fatal contradiction. Gödel made use of the Fundamental Theorem of Arithmetic[5] to ensure the uniqueness of the map from formulae of a system to the natural numbers. We may now expand our diagram to include this part of the allegory:




The final parallel idea is represented by the fact that we are capable of thoughts about marks on a piece of paper. Thus we look at words and sentences from the level of thoughts and ideas. Similarly, the branch of Mathematics known as metamathematics involves looking at statements in a formal system from outside the system. Examples of metastatements might be 'such and such a Principia formula contains three occurrences of such and such a variable' or 'such and such a sequence of formulae in Principia is a demonstration of the last member of the sequence.'[6]

Just as Gödel numbering relates formulae in Principia to numbers, it takes metamathematical statements about formulae and makes them into statements about (Gödel) numbers; that is statements within number theory. Thus the sentences in the previous paragraph might become with the aid of arithmetization: 'The number M has the number N as its factor exactly three times' and 'such and such a sequence of numbers constitutes a factorization of the number L.' Here L, M, and N each represent a number which is the Godel number of the formula they represent.

In our allegory, Godelian ideas would be translated into human ideas via translating the structure into word symbols which are then read and understood by humans. This now completes the skeleton diagram of the form of the proof of Gödel, and of our allegorical proof:




In our story Kurt is set for the final blow to the perfection of the computer with the sentence 'The blueprint representing this sentence is not constructable by the Godelians.' For the real Kurt Gödel this trick was a little more difficult in order to withstand the rigors of mathematical proof. He used some fancy substitution in a manner similar to Cantor's diagonal arguement to create the parallel sentence 'The sentence with Gödel number G is not demonstrable in Principia.' Where G is the Gödel number corresponding to the sentence in quotes. Thus Gödel used the same number G in two ways in a substitution along the style of Cantor.[7]

With this final statement the self referential paradox and the proof are nearly complete. We need only follow the consequences of this reasoning to the finish. If the sentence 'The sentence with Gödel number G is not demonstrable in Principia' expressed in terms of the formal language of Principia via Gödel numbering is true then we must conclude that it is not true. Conversly, if it is not demonstrable, then because the corresponding metamathimatical idea is also not true we conclude that G is demonstrable, a paradox. The sentence with Godel number G is demonstrable if and only if its negation is also true. Thus in order for the formal system to be consistent the formula represented by G must be undeciable, it cannot be proven true or false.[8]

Gödel's actual proof involves more than has been mentioned here. He deals with notions of recursive functions, $\omega$-consistency, and other concepts to make the proof rigorous. However, these additions do not alter the basic skeleton presented in this paper. We also need to note that Gödel's theorem holds for any system powerful enough to describe number theory. More limited formal systems such as propositional calculus can be shown to be complete.[9] Gödel's second incompleteness theorem states that the consistency of a sufficiently powerful formal system cannot be proven from within the system itself. The proof follows a pattern similar to the first incompleteness theorem.

This paper will not discuss any of the many philosophical implications of Gödel's Theorems, but we will close by posing the question of the relation the Godelian race in our story might have to the question of artificial intelligence.

ENDNOTES

[1]The original was published in German under the title "Über Formal Unentscheidbare Sätze der Principia Mathimatica und Verwandter Systeme, I."

[2]Gödel, Kurt. On Formally Undecidable Propositions in Principia Mathematica and Related Systems I. Trans. B. Meltzer. New York, Basic Books 1962. p.57

[3]Cohen, Paul J. Set Theory and Consistency of the Continuum Hypothesis. New York, W. A. Benjamin 1966. p. 3.

[4]Hofstadter, Douglas R. Gödel, Escher, Bach: An Eternal Golden Braid. New York, Vintage Books 1980. p. 438

[5]The Fundamental Theorem of Arithmetic states that every natural number has a unique prime decomposition.

[6]Fitzpatrick, P. J. "To Gödel via Babel." Mind 75(1966) p.332

[7]Hofstadter. p. 446

[8]Nagel, Ernst and James R. Newman. Gödel's Proof. New York, N. Y. Univ. Press 1958. p. 85

[9]Kleene, S. C. Introduction to Metamathematics. New York, D. Van Nostrand Co. 1952. p. 135.

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