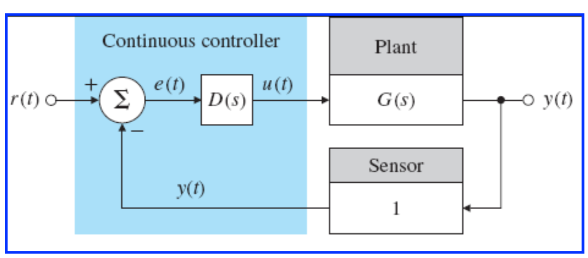
ME 433 – STATE SPACE CONTROL

Lecture 7

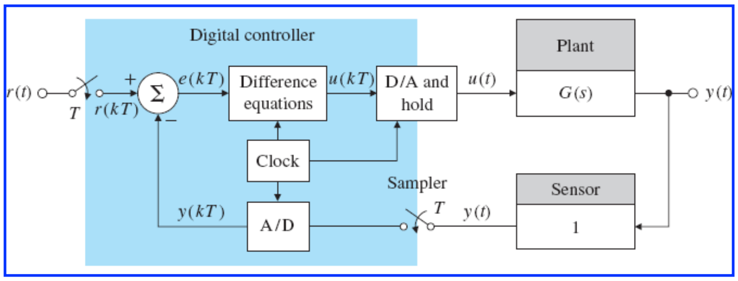
Time Discretization

Continuous-time Feedback System



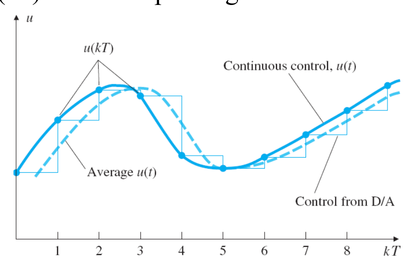
Time Discretization

Discrete-time Feedback System



$r(kT)$, $e(kT)$, $u(kT)$, $y(kT)$: sampled signals

T: sample period
$1/T$: sample rate (Hz)
A/D: analog-to-digital converter
D/A: digital-to-analog converter

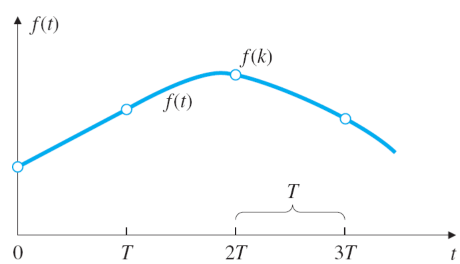


Time Discretization

Function *f(kT)* of time:

$f(k)=f(kT)$ is the sampled version of $f(t)$, and $k=0, 1, 2, 3, \ldots$ refer to the discrete sample times $t_0=0$, $t_1=T$, $t_2=2T$, $t_3=3T$, …

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$



"Differentiation" property:

$$Z\{f(k-1)\} = z^{-1} F(z)$$

Z-Transform

L-Transform	Waveform	Z-Transform
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+\alpha}$	$e^{-\alpha kT}$	$\frac{z}{z-e^{-\alpha T}}$
$\frac{1}{(s+\alpha)^2}$	$kTe^{-\alpha kT}$	$\frac{Tze^{-\alpha T}}{(z-e^{-\alpha T})^2}$
$\frac{\beta}{s^2+\beta^2}$	$\sin(\beta kT)$	$\frac{z\sin(\beta T)}{z^2-2\cos(\beta T)z+1}$
$\frac{s}{s^2+\beta^2}$	$\cos(\beta kT)$	$\frac{z(z-\cos(\beta T))}{z^2-2\cos(\beta T)z+1}$

Z-Transform

Discrete-time System

$$y[k]+a_{k-1}y[k-1]+\cdots+a_n y[k-n]=b_o u[k]+\cdots+b_m u[k-m]$$

Z Transform

$$\left(1+a_1 z+\cdots+a_{n-1}z^{n-1}+a_n z^{-n}\right)Y(s)=\left(b_o+\cdots+b_m z^{-m}\right)U(s)$$

$$T(z)=\frac{Y(z)}{U(z)}=\frac{b_o+\cdots+b_m z^{-m}}{1+a_1 z+\cdots+a_{n-1}z^{n-1}+a_n z^{-n}}$$

Z-Transform

Continuous-time System:

$$f(t) = e^{-\alpha t} \longrightarrow \frac{1}{s+\alpha}$$

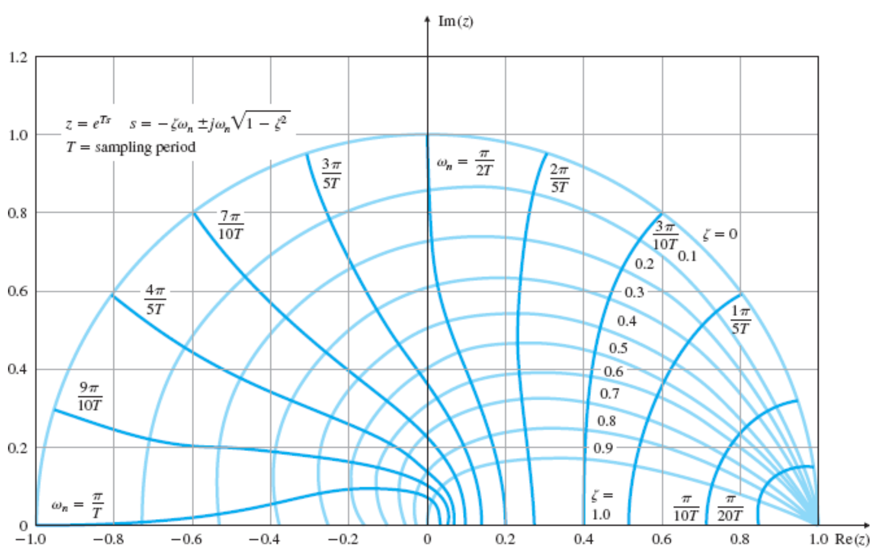
Discrete-time System:

$$f(kT) = e^{-\alpha kT} \longrightarrow \frac{z}{z - e^{-\alpha T}}$$

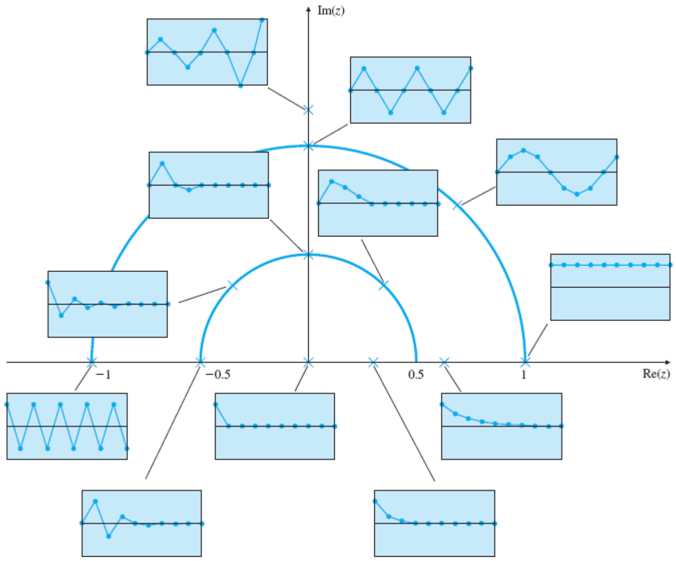
The pole at $s=-\alpha$ in the s-domain corresponds to a pole at $z=e^{-\alpha T}$ in the z-domain:

$$\boxed{z = e^{sT}}$$

Z-Transform

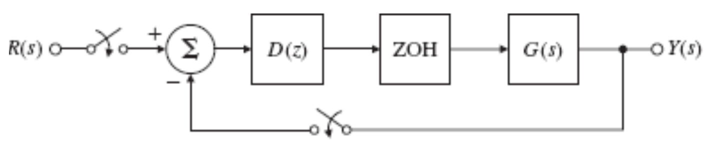


Z-Transform

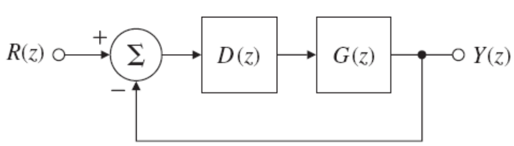


Control Design

Emulation (Discrete Equivalent):

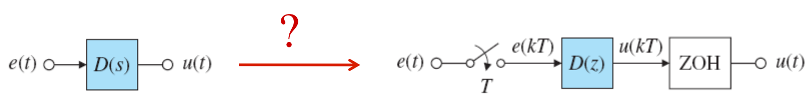


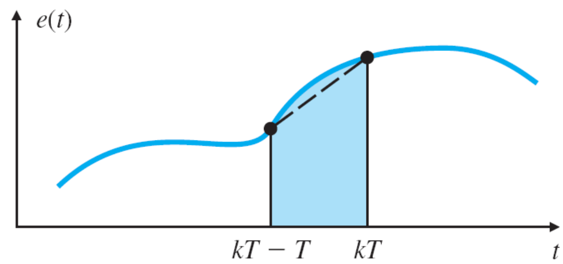
Discrete Design:



Discrete Equivalent (Emulation) Design

Tustin's (Bilinear) Method:





$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Discrete Equivalent (Emulation) Design

Nyquist Theorem:

The Nyquist theorem states that a signal must be sampled at least twice as fast as the bandwidth of the signal to accurately reconstruct the waveform; otherwise, the high-frequency content will *alias* at a frequency inside the spectrum of interest (passband). An alias is a false lower frequency component that appears in sampled data acquired at too low a sampling rate.

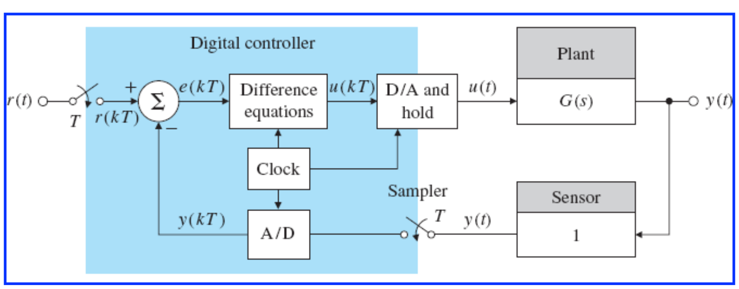
An analog anti-alias filter is often placed between the sensor and the A/D converter. Its function is to reduce the higher-frequency noise components in the analog signal in order to prevent aliasing, that is, having noise or high-frequency components being modulated to a lower frequency by the sampling process.

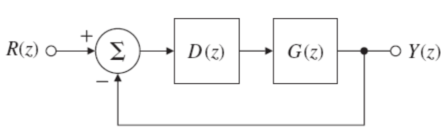
If designing by discrete equivalents, a minimum sample rate of 20 times the bandwidth is recommended. Typically, even faster sampling is useful for best performance. Computational delay should be less than $T/10$.

Discrete Equivalent (Emulation) Design

Example:

Discrete Design





What is the TF between $u(kT)$ and $y(kT)$?

$$G(z) = \left(1 - z^{-1}\right) Z\left\{G(s)/s\right\}$$

Discrete Design

Example:

Solution of State Equation

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

The overall solution for the state equation can be written as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

and the system output as

$$y(t) = Cx(t) + Du(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda + Du(t)$$

Solution of State Equation

Let us now consider $t_0=kT$ and $t=kT+T$, then

$$x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\lambda)}Bu(\lambda)d\lambda$$

Let us also consider

$$u(t) = u(kT) \qquad \forall t \in [kT, kT+T]$$

Using the change of variable $\eta=kT+T-\lambda$, we obtain

$$x(kT+T) = e^{AT}x(kT) + \int_{0}^{T} e^{A\eta}d\eta Bu(kT)$$

Solution of State Equation

Defining

$$\Phi = e^{AT}, \Gamma = \int_{0}^{T} e^{A\eta}d\eta B$$

we can write

$$x(k+1) = \Phi x(k) + \Gamma u(k),$$
$$y(k) = Cx(k) + Du(k).$$

Note that we can write:

$$\Phi = I + AT\Psi$$

$$\Psi = I + \frac{AT}{2!} + \frac{A^2T^2}{3!} + \cdots$$

$$\Gamma = \sum_{k=0}^{\infty} \frac{A^kT^{k+1}}{(k+1)!}B = \sum_{k=0}^{\infty} \frac{A^kT^k}{(k+1)!}TB = \Psi TB$$

Solution of State Equation

We consider the linear, time-invariant system

$$x(k+1) = \Phi x(k) + \Gamma u(k),$$
$$y(k) = Cx(k) + Du(k).$$

We Z transform the state equation to obtain

$$zX(z) = \Phi X(z) + \Gamma U(z)$$

$$X(z) = Z[x(k)], \quad U(z) = Z[u(k)]$$

And we solve to obtain

$$\frac{Y(z)}{U(z)} = C(zI - \Phi)^{-1}\Gamma + D$$