2025 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

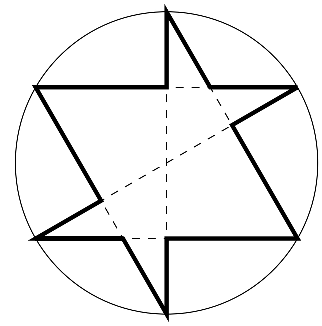
1. Problems

Annotated with number of correct responses out of 106.

1. [2 pts] [100] Points A and B are connected by a 10-mile trail. There is a water fountain on the trail at point C. A group of 8 hikers walked from point A to C, while a group of 12 hikers walked from B to C. The sum of the distances traveled by all hikers in the group that started at A was equal to the sum of the distances traveled by all hikers in the group that started at B. How many miles was the distance from A to C?
2. [2 pts] [93] List all solutions of $x^2 + |x| - 6 = 0$.
3. [2 pts] [87] An isosceles right triangle is inscribed in a circle of radius 1. There are three regions inside the circle but outside the triangle. Two of them are equal. What is the area of one of these equal areas?
4. [2 pts] [76] How many arrangements of the letters of CYCLIC have the property that the C's are evenly spaced; i.e., the number of letters between the first two C's equals the number between the last two?
5. [2 pts] [51] If a, b, c, and d are four **distinct** integers satisfying $\min(a, b) = 3$, $\min(b, c) = 0$, $\max(a, c) = 3$, and $\max(c, d) = 6$, what is the fourth smallest possible value of $a + b + c + d$?
6. [3 pts] [74] If $f(x)$ is a polynomial satisfying $2f(x) + f(1-x) = x^3$, what is the sum of the coefficients of $f(x)$?
7. [3 pts] [49] Consider 5 points in the plane, no three of which are collinear. Let n be the number of circles that can be drawn through at least three of the points. What are the possible values of n?
8. [3 pts] [75] Simplify $\dfrac{2041^3 - 2025^3 - 16^3}{2041 \cdot 2025}$.

9. [3 pts] [54] If $\sin x + \sin^2 x + \sin^3 x + \cdots = 4$, list all possible values of $\cos x + \cos^2 x + \cos^3 x + \cdots$.
10. [3 pts] [74] Six vertex points are evenly spaced around a circle of radius 2, and six line segments connecting pairs of them are drawn as indicated in Figure 1.1. What is the area inside the 12-sided polygon bounded by bold lines in the figure?

Figure 1.1. Find the area inside.



11. [4 pts] [40] Let $f(n) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ and $g(n) = f(n) - f(n-1)$. For how many values of n from 1 to 100 is $g(n)$ even?
12. [4 pts] [56] What is the area of the set of points in the plane satisfying $|x| + |y| + |x+y| \le 1000$?
13. [4 pts] [29] A bug moves on the surface of a cube of side length 2. It is attached by a leash of length 2 to a pole at the center of one face of the cube. What is the area of the region on the cube which the bug can access?
14. [4 pts.] [5] Evaluate $\sum_{i=0}^{100} \lfloor i^{3/2} \rfloor + \sum_{i=0}^{1000} \lfloor i^{2/3} \rfloor$.
15. [4 pts] [10.5] Alice and Bob have identical decks with 3 red cards, 3 white cards, and 3 blue cards. They take turns choosing a random card from their deck, without replacement. Alice goes first. What is the probability that Alice chooses all her red cards before Bob has chosen any red cards?

16. [5 pts] [22] Circles of positive integer radius a, b, and c with $c < b < a$ are mutually tangent, and are tangent to the same line at distinct points. Find (a, b, c) for which c is as small as possible, and a and b as small as possible for this c.

17. [5 pts] [27] Compute the nonzero real triple (a, b, c) such that
$$(x^2 + 2ax + b)^2 + 2a(x^2 + 2ax + b) - b = (x - c)^4$$
is an identity of polynomials.

18. [5 pts] [20] List all positive integers n such that the sum of all positive integers that are $\leq n$ and relatively prime to n equals $2n$.

19. [5 pts] [25] What is the 500th smallest positive integer whose base-3 expansion has no consecutive equal digits? For example, the fourth smallest such number is 5, since the base-3 expansions of the first five positive integers are 1, 2, 10, 11, and 12.

20. [5 pts] [16] Given that 641 is a prime divisor of $2^{32} + 1$, find the smallest positive integer k such that $(2^{16} + 1)^k \equiv 1 \bmod 641$.

21. [6 pts] [8] Triangle ABC has sides of length 5, 6, and 7. Point P lies inside the triangle with angles PAB, PBC, and PCA each equal to α. Find $\tan \alpha$.

22. [6 pts] [17] An integer randomly selected from the interval $[1, 5]$ is written on the board. At each step, the number x on the board is replaced by an integer chosen randomly from the interval
$$\begin{cases} [1, x-1] & \text{if } x > 1 \\ [1, 5] & \text{if } x = 1. \end{cases}$$
Let $P_n(x)$ denote the probability that the number on the board after n steps is x. Find $\lim_{n \to \infty} P_n(2)$.

23. [6 pts] [7] Let ABC be a triangle with circumcenter O and $\angle B$ obtuse. Suppose the tangent to its circumcircle at A intersects ray CB at D, and that $AO = AD = 2$. Suppose that F lies on BC with $AF \perp OD$, and let E be the point where AF meets OD. Let H be the point where the circle passing through ADO meets BC. What is the area of triangle OEF if $FH = \frac{\sqrt{3}}{3}$?

24. [6 pts] [4] Four distinct points are selected randomly from inside a circle. These determine four triangles (including the possibility of degenerate triangles). Then two of the four triangles are randomly selected. What is the probability that the center of the circle lies in the union of the interiors of the two triangles selected?

25. [6 pts] [5] List all positive integers n for which there is a convex n-gon whose angles, in degrees, are distinct integers in arithmetic progression. List them in order in concatenated form; i.e., not separated by commas. For example, the correct answer begins 345.

2. Solutions

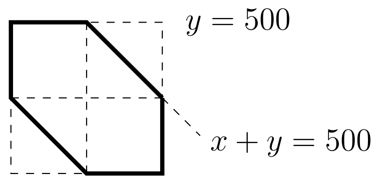
1. 6. If x is the distance from A to C, then $8x = 12(10 - x)$ so $20x = 120$ and $x = 6$.
2. ± 2. The solutions with $x > 0$ must satisfy $x > 0$ and $x^2+x-6 = 0$ so just $x = 2$. But $-x$ satisfies the given equation if and only if x does.
3. $\frac{\pi}{4} - \frac{1}{2}$. The hypotenuse of the triangle is a diameter of the circle. Thus the desired area is $\frac{1}{2}(A_S - A_T)$, where A_S (resp. A_T) is the area of the semicircle (resp. the triangle). The answer is $\frac{1}{2}(\frac{\pi}{2} - 1)$, since the legs of the triangle have length $\sqrt{2}$.
4. 36. Letting x represent the other letters, the arrangements are CCCxxx, xCCCxx, xxCCCx, xxxCCC, CxCxCx, and xCxCxC. For each, there are 6 ways to arrange the other letters.
5. 17. Since $b > 0$, we must have $c = 0$. Then $a = 3$ and $d = 6$. Since the numbers are distinct, we must have $b > 3$ and $b \neq 6$. Then $a + c + d = 9$, while $b = 4$, 5, 7, 8.
6. $\frac{2}{3}$. Letting $x = 1$ and then 0, we have $2f(1) + f(0) = 1$ and $2f(0) + f(1) = 0$. Solving yields $f(1) = \frac{2}{3}$, and this is the sum of the coefficients of f.
7. 1, 7, 10. If all five lie on a circle, $n = 1$. If no four lie on a circle, then $n = \binom{5}{3} = 10$. If four lie on a circle with the fifth one not on it, then $n = 1 + \binom{4}{2} = 7$.
8. 48. If divided by 16, it would be, with $a = 2025$ and $b = 16$,
$$\frac{(a+b)^3 - a^3 - b^3}{(a+b)ab} = \frac{3a^2b + 3ab^2}{a^2b + ab^2} = 3.$$
9. $\frac{3}{2}$ and $-\frac{3}{8}$. We have $\frac{\sin x}{1-\sin x} = 4$, so $\sin x = \frac{4}{5}$, hence $\cos x = \pm\frac{3}{5}$ and $\frac{\cos x}{1-\cos x} = \frac{3/5}{2/5}$ or $\frac{-3/5}{1+\frac{3}{5}}$.

10. $\frac{10}{3}\sqrt{3}$. Three of the vertices are $A = (0, 2)$, $B = (\sqrt{3}, 1)$, and $C = (\sqrt{3}, -1)$. The right half consists of a right triangle with hypotenuse AC and a small right triangle with vertex at B. The first triangle has base $\sqrt{3}$ and height 3, while the second is congruent to a small triangle with vertex at A with sides 1/3 times those of the first triangle. Thus the desired area is $2(\frac{1}{2}\sqrt{3} \cdot 3(1 + \frac{1}{9}))$.

11. 90. $\lfloor \frac{n}{d} \rfloor - \lfloor \frac{n-1}{d} \rfloor$ equals 1 if d divides n and 0 otherwise, so $g(n)$ is the number of positive divisors of n. This is odd iff n is a perfect square. Since there are 10 perfect squares ≤ 100, the answer is 90.

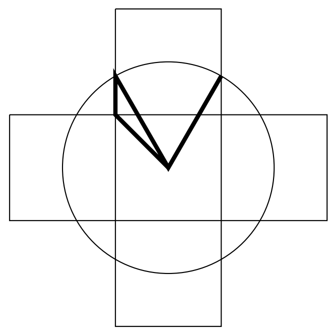
12. 750000. It is the region inside the square $|x| \leq 500$, $|y| \leq 500$ satisfying also $|x + y| \leq 500$, so the area is $\frac{3}{4} \cdot 1000^2$. See Figure 2.1.

Figure 2.1. Region for problem 12



13. $\frac{8\pi}{3}+4\sqrt{3}-4$. The area is the intersection of the unfolded cube with a circle of radius 2. See Figure 2.2. The desired area is 4 times that of the indicated sector plus 8 times that of the indicated triangle. The area of the sector is $\frac{1}{6}\pi 2^2$. The triangle has base $\sqrt{3}-1$ and height 1.

Figure 2.2. Region for problem 13

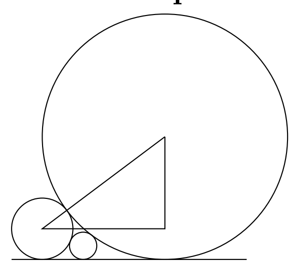


14. 100010. The first sum is the number of lattice points with $y > 0$ on or below the curve $y = x^{3/2}$ for $0 < x \le 100$. By symmetry, the second sum can be considered to be the number of lattice points with $x > 0$ to the left of that curve for $0 < y \le 1000$. The total number of points is 100×1000 plus the number of lattice points on the curve, since they appear twice in the sum. The answer is $100000 + 10$.

15. 5/168. Suppose each of them choose all 9 of their cards in a random order, and consider the positions of the 3 red cards for each of them. There are $(\binom{9}{3})^2$ possibilities. Consider the 10-card sequence consisting of Alice's cards until she gets her third red, followed by Bob's cards starting with the card that he would have played after Alice's last red card. There must be 6 reds in this sequence, and this is what we are enumerating. The probability is $\binom{10}{6}/(\binom{9}{6})^2$, which simplifies to $5/168$.

16. $(36, 9, 4)$. In Figure 2.3, the radii are b, c, and a. The hypotenuse is $a+b$ and vertical side is $a-b$, so the horizontal side is $2\sqrt{ab}$. Similarly the distance between other points of tangency are $2\sqrt{bc}$ and $2\sqrt{ac}$. Thus $\sqrt{ab} = \sqrt{ac} + \sqrt{bc}$. Squaring yields $c = ab/(a + b + 2\sqrt{ab})$. Since c is an integer, $\sqrt{ab}$ must be an integer, and so $a = k\alpha^2$ and $b = k\beta^2$ for integers k, α, and β, with α and β relatively prime. Thus $c = k\alpha^2\beta^2/(\alpha+\beta)^2$ and is minimized with $k = (\alpha + \beta)^2$, $\beta = 1$, and $\alpha = 2$.

Figure 2.3. Three circles for problem 16

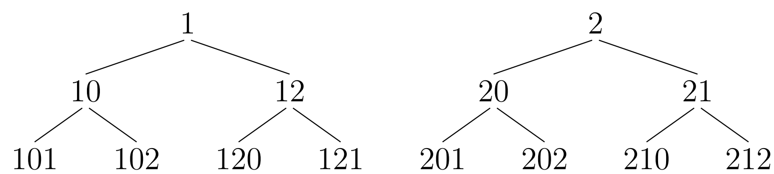


17. $(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{2})$. Let $P(x) = x^2 + 2ax + b$ and $Q(x) = x^2 + 2ax - b$. Then we are looking for nonzero a and b such that $Q(P(x))$ has only a single repeated root. The roots of $Q(P(x))$ are the solutions of $P(x) = r_1$ and $P(x) = r_2$, where r_1 and r_2 are the roots of Q. Thus Q must have a repeated root, hence its discriminant $4a^2 + 4b$ must equal 0, and its root is $-a$. Now $P(x) = -a$ has a repeated solution, and so the discriminant of $P(x) + a$ must equal 0. We obtain $4a^2 - 4(a + b) = 0$, and using $b = -a^2$ we obtain $8a^2 - 4a = 0$, so the nonzero solution has $a = \frac{1}{2}$ and $b = -\frac{1}{4}$. Finally, c is the solution of $P(x) + a = 0$, which is $-a = -\frac{1}{2}$.

18. 5, 8, 10, and 12. For $n > 2$, positive integers k relatively prime to n come in pairs $\{k, n-k\}$ which sum to n. Thus the sum of integers relatively prime to n equals $\frac{n}{2}\phi(n)$, where $\phi(n)$ is the number of positive integers $\le n$ relatively prime to n. Thus we want integers for which $\phi(n) = 4$. Since $\phi(\prod p_i^{\alpha_i}) = \prod p_i^{\alpha_i - 1}(p_i - 1)$, it is easy to see that $\phi(n) = 4$ iff $n = 5$, 2^3, $2 \cdot 5$, or $2^2 \cdot 3$.

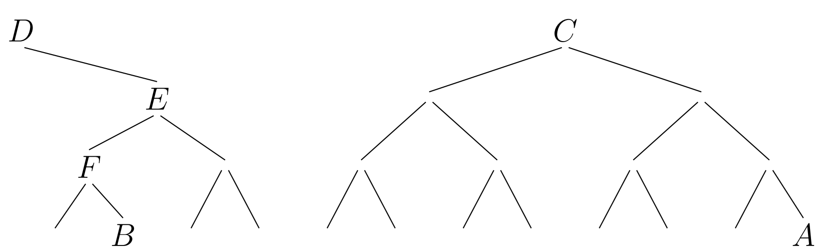
19. 5690. Form a table of numbers of the desired type beginning as in Figure 2.4. Below each number are the two ways of appending a digit to it.

Figure 2.4. The first 14 desired base-3 expansions



Continuing in this manner gives all the desired numbers in order. After 8 rows, we will have the first $2^9 - 2 = 510$. In Figure 2.5, we draw the lower right portion of the first 8 rows. Node A is 21212121, the 510th desired number. So we want the number at node B, the $(510-10)$th desired number. Node C is 21212, so nodes D, E, F, and B are, respectively, 21210, 212102, 2121020, and 21210202. The integer for this is $2 \cdot 2187 + 729 + 2 \cdot 243 + 81 + 2 \cdot 9 + 2$.

Figure 2.5. Bottom right portion of first 8 rows



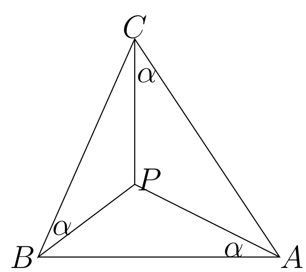
20. 128. Let $n = 2^{16}+1$. Working mod 641, we have $(n-1)^2+1 \equiv 0$ and so $n^2 \equiv 2n-2$. Thus $n^4 \equiv 4(n^2-2n+1) \equiv -4$. Thus $n^{128} \equiv (-4)^{32} = (2^{32})^2 \equiv (-1)^2 = 1$. If $n^{64} \equiv 1$, then $(-4)^{16} \equiv 1$, so $2^{32} \equiv 1$, contradicting $2^{32} \equiv -1$.

21. $\frac{12}{55}\sqrt{6}$. Let $AB = 5$, $BC = 6$, and $CA = 7$, and $AP = x$, $BP = y$, and $CP = z$. See Figure 2.6. Then

$$y^2 = 5^2 + x^2 - 10x\cos\alpha$$
$$z^2 = 6^2 + y^2 - 12y\cos\alpha$$
$$x^2 = 7^2 + z^2 - 14z\cos\alpha.$$

Adding these gives $(5x+6y+7z)\cos\alpha = 55$. Computing the area of triangle ABC both by Heron's formula and by summing the areas of three subtriangles yields $\sqrt{9\cdot 2\cdot 3\cdot 4} = \frac{1}{2}(5x+6y+7z)\sin\alpha$. Dividing the two equations yields $\tan\alpha = \frac{12}{55}\sqrt{6}$.

Figure 2.6. Triangle for problem 21



22. $30/137$. Let $f(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} P_{n-1}(x)$. The only way to get 5 is if 1 is on the board. Therefore $P_n(5) = \frac{1}{5}P_{n-1}(1)$ and so $f(5) = \frac{1}{5}f(1)$. The only ways to get a 4 are from 1 or 5, so $P_n(4) = \frac{1}{5}P_{n-1}(1) + \frac{1}{4}P_{n-1}(5)$ and

$$f(4) = \tfrac{1}{5}f(1) + \tfrac{1}{4}f(5) = (\tfrac{1}{5} + \tfrac{1}{5}\cdot\tfrac{1}{4})f(1) = \tfrac{1}{4}f(1).$$

Similarly $P_n(3) = \frac{1}{5}P_{n-1}(1) + \frac{1}{4}P_{n-1}(5) + \frac{1}{3}P_{n-1}(4)$ and

$$f(3) = \tfrac{1}{5}f(1) + \tfrac{1}{4}f(5) + \tfrac{1}{3}f(4) = \tfrac{1}{4}f(1) + \tfrac{1}{3}\cdot\tfrac{1}{4}f(1) = \tfrac{1}{3}f(1).$$

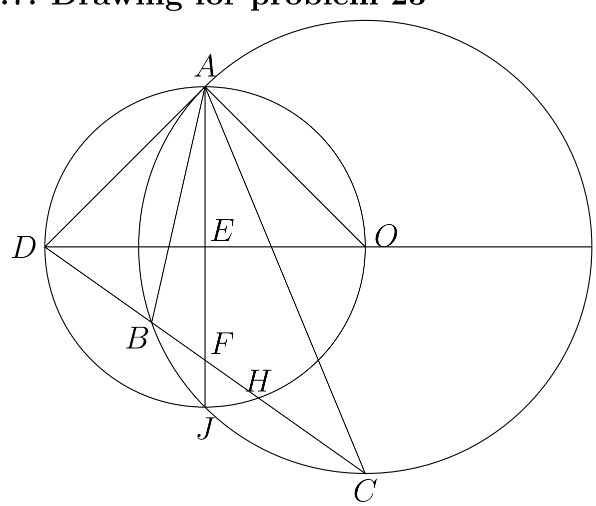
Similarly $f(2) = \frac{1}{2}f(1)$. Thus

$$1 = f(1)+f(2)+f(3)+f(4)+f(5) = (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5})f(1) = \tfrac{137}{60}f(1).$$

Therefore $f(1) = \frac{60}{137}$ and $f(2) = \frac{30}{137}$.

23. $\frac{1}{2}\sqrt{2}$. Let J be the point where the extension of AF meets circle ADO. Let $x = DF$ and $y = EF$. Power of a point using chords DH and AJ yields $x\frac{\sqrt{3}}{3} = (\sqrt{2}+y)(\sqrt{2}-y)$ while Pythagoras on triangle DEF yields $x^2 = y^2 + 2$. Combining, we get $x\frac{\sqrt{3}}{3} = 2 - (x^2 - 2)$ so $x^2 + x\frac{\sqrt{3}}{3} - 4 = 0$. The solution is $x = \sqrt{3}$ and $y = 1$, so the area of the right triangle OEF is $\frac{1}{2}\sqrt{2}\cdot 1$.

Figure 2.7. Drawing for problem 23



24. 5/12. We first calculate the probability that the center is in the union of the four triangles. The center does not lie in the union of the four triangles iff the four points lie on the same side of some diameter of the circle iff this is true of their radial projections onto the circle iff for one of these, the other three are within 180 degrees clockwise of the point, and such a point is essentially unique. Given a point, the probability that the other three lie in this arc is $(\frac{1}{2})^3 = \frac{1}{8}$, so the probability that the center does not lie in the convex hull is $4 \cdot \frac{1}{8} = \frac{1}{2}$.

    Ignoring possibilities with probability 0, the center lies in the union of the four triangles iff it lies in the intersection of two of the triangles, and these possibilities are disjoint. Since there are six pairs, the probability of the center lying in a randomly selected intersection is $\frac{1}{2}/6 = \frac{1}{12}$. The center is in the union of the two selected triangles iff it is in the union of the four but not in the intersection of the two which we did not select. This probability is $\frac{1}{2} - \frac{1}{12}$.

25. 3456891012151618. The average angle is $\frac{(n-2)180}{n}$. Let L (resp. S) denote the largest (resp. smallest) angle. Then

$$179 \geq L = \tfrac{L+S}{2} + \tfrac{L-S}{2} \geq \tfrac{(n-2)180}{n} + \tfrac{n-1}{2},$$

    which simplifies to $720 \geq n(n+1)$, so $n \leq 26$. Since the angles are in arithmetic progression, twice the average angle is an integer, which implies that n divides 720. This leaves 13 possibilities. If n is even, the average angle cannot be an angle, and if also $\frac{(n-2)180}{n}$ is an integer, then the common difference cannot be 1. This applies to $n = 20$ and 24, for which the average angles are 162 and 165, resp. This will make the largest angle > 180 in each case. We list the eleven solutions, with (n, S, d), where S is the smallest angle, and d the common difference. (3,59,1), (4,87,2), (5,106,1), (6,115,2), (8,128,2), (9,136,1), (10,135,2), (12,139,2), (15,149,1), (16,150,1), (18,143,2).