THE CONNECTIVE KO-THEORY OF THE EILENBERG-MACLANE SPACE $K(\mathbb{Z}_2, 2)$, I: THE E_2 PAGE

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Abstract. We compute the E_2 page of the Adams spectral sequence converging to the connective KO-theory of the second mod 2 Eilenberg-MacLane space, $ko_*(K(\mathbb{Z}/2, 2))$. This required a careful analysis of the structure of $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}_2)$ as a module over the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. Complete analysis of the spectral sequence will be performed in a subsequent paper.

1. Introduction

Let $\mathbb{Z}_2 = \mathbb{Z}/2$ and let K_2 denote the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$. In [8], the authors gave a complete determination of the connective complex K-theory groups $ku_*(K_2)$ and $ku^*(K_2)$. The original motivation for this work was from [14] and [9], which studied Stiefel-Whitney classes of Spin manifolds. Because of the relationship ([2]) of the Spin cobordism spectrum and the spectrum ko for connective real K-theory, information about $ko_*(K(\mathbb{Z}_2, n))$ gave useful results about Spin manifolds. For complete calculations the authors were led to the more tractable ku groups. In this paper, we return to the ko groups.

We give a complete determination of the E_2 page of the Adams spectral sequences (ASS) converging to $ko_*(K_2)$ and $ko^*(K_2)$. In a subsequent paper, we will complete the calculation by determining the differentials and extensions in the spectral sequences. We choose to split this E_2 work off because we feel that it involves some clever arguments that we would not want to have obscured in a paper with massive ASS charts.

Most of our focus will be on the homology groups $ko_*(K_2)$, in part because of its connection with the motivating problem and in part because its ASS is of a more

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familiar form than that for $ko^*(-)$. In [8], most of the work was done for the cohomology groups $ku^*(K_2)$, largely because of the product structure. That structure, along with a comparison with the mod-p groups $k(1)^*(K_2)$, enabled us to find the differentials in the spectral sequence for $ku^*(K_2)$, and we can use that information to deduce differentials in the other spectral sequences. Similarly to the situation for ku in [8], the ko-homology and ko-cohomology groups of K_2 are Pontryagin duals of one another. We discuss this in Section 4.

Let A_1 denote the subalgebra of the mod-2 Steenrod algebra generated by Sq^1 and Sq^2, and let E_1 denote the exterior subalgebra generated by the Milnor primitives $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. The ASS converging to $ko_*(X)$ has $E_2^{s,t} = \mathrm{Ext}_{A_1}^{s,t}(H^*X, \mathbb{Z}_2)$, while that for $ku_*(X)$ has $E_2 = \mathrm{Ext}_{E_1}(H^*X, \mathbb{Z}_2)$. All cohomology groups have coefficients in $\mathbb{Z}_2$. The first step toward $ku^*(K_2)$ was finding a splitting of H^*K_2 as a direct sum of reduced E_1-modules ([8, Proposition 2.11 and (2.16)]). (A *reduced* module is one containing no free submodules.) In Section 3, we describe a corresponding splitting as A_1-modules (Theorem 3.9) and the groups $\mathrm{Ext}_{A_1}(-, \mathbb{Z}_2)$ for all of the summands. This then will be the E_2 page of the ASS, the main result of this paper.

2. The A_1-summands M_k

An important part of the E_1 splitting of H^*K_2 was a family of E_1-modules M_k for $k \geq 4$ ([8, (2.13), (2.14), (2.15)]). In this section, we find corresponding A_1-modules, which we also call M_k. Although the structure of these A_1-modules as E_1-modules is very similar to that of the corresponding E_1-modules of [8] (in fact isomorphic if $k \equiv 0, 1 \bmod 4$), finding classes with the correct Sq^2 behavior was a nontrivial task.

Let u_0 denote the nonzero element of $H^2(K_2)$, and define u_j inductively by $u_{j+1} = \mathrm{Sq}^{2^j} u_j$. Then $H^*(K_2) = \mathbb{Z}_2[u_j : j \geq 0]$ with $|u_j| = 2^j + 1$. Let $S = (\mathrm{Sq}^1, \mathrm{Sq}^2)$. One easily checks that

$$S(u_j) = \begin{cases} (u_1, u_0^2) & j = 0 \\ (0, u_2) & j = 1 \\ (u_{j-1}^2, 0) & j \geq 2. \end{cases}$$

In Lemma 2.1 we replace u_j with generators x_j for $j \geq 4$ with similar properties except that $\mathrm{Sq}^2\mathrm{Sq}^1(x_4) = 0$.

Lemma 2.1. *There are elements $x_j \in H^{2^j+1}(K_2)$ for $j \geq 4$ satisfying*

1. $x_j \equiv u_j$ *mod decomposables,*
2. $\mathrm{Sq}^1(x_4) = c_{18} \neq 0$, $\mathrm{Sq}^2(c_{18}) = 0$, $\mathrm{Sq}^2(x_4) = 0$, *and*
3. $S(x_j) = (x_{j-1}^2, 0)$ *for* $j \geq 5$.

Proof. We first introduce an intermediate set of generators w_j defined by

$$w_j = \begin{cases} u_j & j = 0, 1 \\ u_0 u_1 + u_2 & j = 2 \\ u_1^{2^{j-2}} u_{j-2} + u_0^{2^{j-2}} u_{j-1} + u_j & j \geq 3. \end{cases}$$

These satisfy

$$S(w_j) = \begin{cases} (w_1, w_0^2) & j = 0 \\ (0, w_2 + w_0 w_1) & j = 1 \\ (0, w_0 w_2) & j = 2 \\ (w_{j-1}^2, 0) & j \geq 3. \end{cases}$$

Now we define $x_4 = w_0 w_2^3 + w_4$ and, for $j \geq 5$

$$x_j = w_0^{2^{j-4}} w_2^{2^{j-3}} w_{j-2} + w_1^{3 \cdot 2^{j-5}} w_2^{2^{j-4}} w_3^{2^{j-5}} w_{j-3} + w_0^{2^{j-5}} w_1^{2^{j-4}} w_2^{2^{j-3}} w_{j-3} + w_j.$$

One can check that these satisfy the claims of the lemma. ■

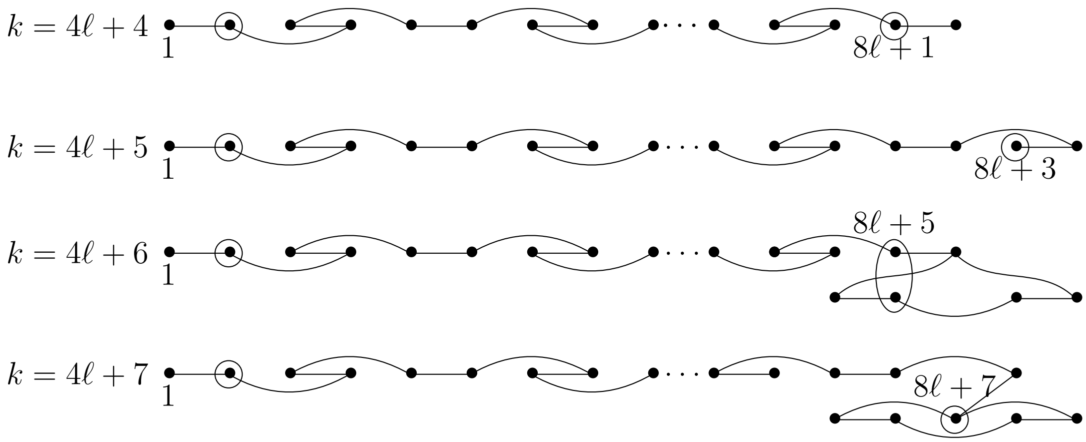
Theorem 2.2. *For $k \geq 4$ there are Q_0-free A_1-submodules $M_k \subset H^*(K_2)$ with*

$$H_*(M_k; Q_1) = \begin{cases} \langle c_{18}, x_4 \rangle & k = 4 \\ \langle x_{k-1}^2, c_{18} x_4 \prod_{t=4}^{k-2} x_t^2 \rangle & k \geq 5. \end{cases}$$

The A_1-module $\Sigma^{-2^k} M_k$ has the form in Figure 2.3.

Here and throughout $\langle s_1, \ldots, s_k \rangle$ denotes the span (resp. graded span) of elements in a vector space (resp. graded vector space). We depict A_1-modules with straight segments showing Sq^1, and curved segments Sq^2. We circle the Q_1-homology classes.

Figure 2.3. Modules $\Sigma^{-2^k} M_k$.



For example, if $k = 4\ell + 4$, $\Sigma^{-2^k} M_k$ has a single nonzero class g_i for $1 \le i \le 8\ell + 2$ with

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2(g_i) = \mathrm{Sq}^1\, g_{i+4} \ne 0 \text{ if } i \equiv 3\ (4),\ i \le 8\ell - 5,$$

and $\mathrm{Sq}^2\,\mathrm{Sq}^1(g_1) = \mathrm{Sq}^1(g_3) \ne 0$.

Proof of Theorem 2.2. We use the classes x_j, $j \ge 4$, of Lemma 2.1, but find it convenient to write c_{18} as x_3^2, even though it isn't a perfect square. In the discussion below we treat it as a perfect square. For $k \ge 4$, let $\mathbb{M}_k$ denote the finite A_1-submodule of H^*K_2 with basis all elements $\prod_{j=3}^{k} x_j^{e_j}$ satisfying $\sum e_j 2^j = 2^k$. Our desired A_1-module M_k will be a submodule of $\mathbb{M}_k$.

We first show that $\mathbb{M}_k$ is Q_0-free. Every monomial in $\mathbb{M}_k$ which is a perfect square can be written uniquely as $\prod_{s\in S} x_s^2 \cdot \prod_{t\in T} x_t^{2e_t}$ with $e_t > 1$ and S and T disjoint. It determines a Q_0-free summand

$$\prod_{t\in T} x_t^{2e_t-2} \bigotimes_{i\in S\cup T} \langle x_{i+1}, x_i^2\rangle.$$

Every monomial in $\mathbb{M}_k$ is in a unique one of these summands, as can be seen by writing the monomial as $P \cdot \prod_{u\in U} x_u$ with P a perfect square. This monomial is in the Q_0-free summand determined as above from $P \cdot \prod_{u\in U} x_{u-1}^2$.

We now show, somewhat similarly, that

$$H_*(\mathbb{M}_k;Q_1)=\langle x_{k-1}^2, x_4\prod_{t=3}^{k-2}x_t^2\rangle.$$

Let $k\ge 5$, as the case $k=4$ is elementary. Every monomial in $\mathbb{M}_k$ which is a perfect square or x_4 times a perfect square can be written uniquely as $\prod_{s\in S}x_s^2\cdot x_4^\varepsilon\cdot\prod_{t\in T}x_t^{2e_t}$ with $e_t\ge 2$, S and T disjoint, and $\varepsilon\in\{0,1\}$. Also, $T\ne\emptyset$ unless the monomial is x_{k-1}^2 or $x_3^2x_4^3x_5^2\cdots x_{k-2}^2$, in order to have $\sum e_j2^j=2^k$. This monomial determines a Q_1-free summand

$$\prod_{s\in S}x_s^2\cdot x_4^\varepsilon\cdot\prod_{t\in T}x_t^{2e_t-4}\bigotimes_{t\in T}\langle x_t^4, x_{t+2}\rangle.$$

Every monomial in $\mathbb{M}_k$ except x_{k-1}^2 and $x_3^2x_4^3x_5^2\cdots x_{k-2}^2$ is in a unique one of these by writing it as

$$P\cdot x_4^\varepsilon\cdot\prod_{\substack{t\in T\\ t>2}}x_{t+2}$$

with P a perfect square; it is in the Q_0-free summand determined as above from $P\cdot x_4^\varepsilon\prod_{t\in T}x_t^4$.

By [12, Proposition 13.13 and p.203], the A_1-module $\mathbb{M}_k$ has an expression, unique up to isomorphism, as $M_k\oplus F$, with F free and M_k reduced. This M_k is Q_0-free and has the Q_1-homology stated in the theorem. To get a sense of why this is true, it is impossible for a Q_0-free module to have just one Q_1-homology class. Thus the two Q_1-homology classes must be in the same summand and what is left must be free over A_1.

We will determine its precise structure.

The module $\mathbb{M}_4$ has only the classes $\langle x_4, x_3^2\rangle$, so this is also M_4. For $k\ge 5$, $\mathbb{M}_k$ in gradings $\le 2^k+4$ has just the classes $\langle x_k, x_{k-1}^2, x_{k-2}^2x_{k-1}, x_{k-2}^4\rangle$, in which Sq^1 and Sq^2 act as depicted on the left four dots in each row of Figure 2.3. We will use Yu's Theorem ([4, Theorem 7.1]) to show that M_k must have the form claimed in the theorem. We thank Bob Bruner for suggesting the use of Yu's Theorem.

For $k\ge 5$, let $\mathbb{M}_k^*$ denote the A_1-module dual to $\mathbb{M}_k$. Its top class x_k^* is in grading -2^k-1 and bottom class $(x_3^{2^{k-3}})^*$ is in grading -2^k-2^{k-3}. Let $(\mathbb{M}_k^*)^+$ denote an A_1-module which agrees with $\mathbb{M}_k^*$ in gradings less than -2^k and for $i\ge -2^k$ has a single nonzero class y_i in grading i, with $\mathrm{Sq}^2\,\mathrm{Sq}^3\,y_{4j} = y_{4j+5} = \mathrm{Sq}^1\,y_{4j+4}$, and $0 \neq \mathrm{Sq}^1\,y_{-2^k} \in \mathrm{im}(\mathrm{Sq}^2)$. This $(\mathbb{M}_k^*)^+$ is Q_0-free and has a single nonzero Q_1-homology class, dual to $x_4 \prod_{t=3}^{k-2} x_t^2$, in grading $7 - 2^k - 2k$. By [12, Proposition 13.13 and p.203], $(\mathcal{M}_k^*)^+$ is isomorphic to the direct sum of a reduced module R and a free module. Since R is Q_0-free and reduced with a single nonzero Q_1-homology class, by Yu's Theorem, R is isomorphic to a shifted version of one of the four modules P_i, $0 \le i \le 3$, depicted in [4, Figure 1]. These modules begin with a form dual to one of the four endings of the modules in Figure 2.3, followed by an infinite string of $\mathrm{Sq}^1\,z_n = \mathrm{Sq}^2\,\mathrm{Sq}^3\,z_{n-4}$.

Our module M_k is defined as the dual of R/T, where T is the submodule of R consisting of classes of grading $\ge -2^k$. This M_k will begin the same way as $\mathbb{M}_k$, as $\Sigma^{2^k}\langle g_1, \mathrm{Sq}^1\,g_1, g_3, \mathrm{Sq}^1\,g_3 = \mathrm{Sq}^2\,\mathrm{Sq}^1\,g_1\rangle$, and will end with one of the four types in Figure 2.3, although *a priori* it could have a different length. Its top Q_1-homology class is in grading $2^k + 2k - 7$.

Since A_1 has 8 basis elements, the total number of basis elements in M_k will be congruent mod 8 to the number in $\mathbb{M}_k$. There is a 1-1 correspondence between a basis for $\mathbb{M}_k$ and the set of partitions of 2^{k-3} into 2-powers. (e_j tells the number of occurrences of 2^{j-3}.) It is proved in [6] that this number of partitions is $\equiv 2$ mod 8 if k is even, and is $\equiv 4$ mod 8 if k is odd.

Let $k = 4\ell + 4$. The first module in Figure 2.3 is the only possibility that satisfies that the top Q_1-homology class is in grading $2^k + 2k - 7 = 2^k + 8\ell + 1$ and the number of basis elements is $\equiv 2$ mod 8. The second and fourth types in Figure 2.3 have their top Q_1-homology class in grading 3 mod 4, while if the third type had its top Q_1-homology class in $2^k + 8\ell + 1$, its number of basis elements would be 6 mod 8. A similar analysis, utilizing top Q_1-homology class mod 4 and number of basis elements mod 8, shows the M_k must be as claimed.

■

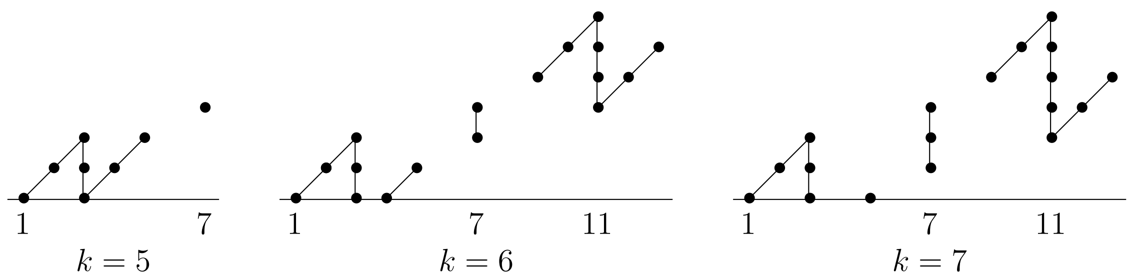
Prior to discovering this proof, we had laboriously found explicit bases for M_k for $k \le 9$. For example, with $abcd$ denoting $x_6^a x_5^b x_4^c c_{18}^d$, the basis for M_7 had x_7, 2000, 1200, 0400, 1040, 0240, 0080 along the top, as pictured in Figure 2.3, and 1111+0320,

$0311 + 1031 + 1102 + 0240$, $0231 + 1022 + 0160$, $0151 + 1013 + 0222 + 0080$, and $0071 + 0142 + 0213 + 1004$ along the bottom.

3. Ext charts and tensor products

There is a nice pattern to the charts $\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{-2^k} M_k, \mathbb{Z}_2)$, depicted, as usual, in coordinates $(t-s, s)$. They are similar to familiar charts of $\mathrm{Ext}_{A_1}(H^*P^{2n}, \mathbb{Z}_2)$ (e.g., [7]). In fact, there are A_1-module isomorphisms $\Sigma^{-2^{4\ell+4}} M_{4\ell+4} \approx H^*P^{8\ell+2}$ and $\Sigma^{-2^{4\ell+5}} M_{4\ell+5} \approx H^*P^{8\ell+4}$. For all k, all classes in these charts are v_1^4-periodic; i.e., $\mathrm{Ext}^{s,t} \to \mathrm{Ext}^{s+4,t+12}$ is bijective for $s \geq 0$. All the charts have the same upper edge, $(8i + x, 4i + y)$ for $(x, y) = (1, 0)$, $(2, 1)$, $(3, 2)$, and $(7, 3)$. The lower edge drops by 1 for each increase in k, as long as $s \geq 0$. In Figure 3.1 we show the beginning of the charts for $5 \leq k \leq 7$. These Ext charts are easily obtained by standard methods from the explicit description of the modules in Figure 2.3. See [5, Appendix A] for a rather detailed discussion of these methods.

Figure 3.1. $\mathrm{Ext}_{A_1}(\Sigma^{-2^k} M_k, \mathbb{Z}_2)$**.**



Explicitly, $\Sigma^{-2^k} M_k$ has, for $i \geq 0$,

- 0 in $8i + 6, 8$,
- $\mathbb{Z}_2$ in $8i + 1, 2$ of filtration $4i + 0, 1$,
- $\mathbb{Z}_2$ in $8i + 4, 5$ of filtration $4i - k + 6, 7$ if $4i - k + 6, 7 \geq 0$, else 0,
- $\mathbb{Z}/2^{k-4}$ in $8i + 7$ with generator of filtration $4i - k + 8$ if $4i - k + 8 \geq 0$, else $\mathbb{Z}/2^{4i+4}$ with generator of filtration 0, and
- $\mathbb{Z}/2^{k-2}$ in $8i + 3$ with generator of filtration $4i - k + 5$ if $4i - k + 5 \geq 0$, else $\mathbb{Z}/2^{4i+3}$ with generator of filtration 0.

Here, as usual, d dots connected by vertical segments yield a $\mathbb{Z}/2^d$ summand.

The A_1-modules M_k in Section 2 correspond to the E_1-modules M_k in the E_1-splitting of $H^*(K_2)$ in [8, (2.16)]. The correspondence is that, as an E_1-module, the A_1-module M_k is isomorphic to the E_1-module M_k plus perhaps a single copy of E_1. Moreover, the Q_1-homology classes agree, with u_j replaced by x_j. Also involved in the E_1 splitting in [8, (2.16)] were summands $M_k \cdot P$, where P is a product of finitely many distinct classes u_j^2 with $j \geq k$. Although u_j^2 is acted on trivially by E_1, $\mathrm{Sq}^2(u_j^2) \neq 0$, so the corresponding A_1 summands must do more than just multiply by the product of the classes u_j^2. To maintain some consistency with [8], in Definition 3.3 we will define $M_k z_j$ to be a reduced Q_0-free A_1-module with

$$H_*(M_k z_j; Q_1) = H_*(M_k; Q_1) \otimes \langle u_{j+1}^2 \rangle, \tag{3.2}$$

and similarly for products with more than one z_j.

For $j \geq 3$, let $G_j = \langle u_{j+2}, u_{j+1}^2, u_j^4 \rangle$ with $\mathrm{Sq}^2 \mathrm{Sq}^1 u_{j+2} = u_j^4$. If M is a Q_0-free A_1-module, then $M \otimes G_j$ is Q_0-free and

$$H_*(M \otimes G_j; Q_1) = H_*(M; Q_1) \otimes \langle u_{j+1}^2 \rangle.$$

Definition 3.3. *We define $M_k z_j$ to be the reduced summand of the A_1-module $M_k \otimes G_j$.*

Let P_j be the A_1-module for which there is a short exact sequence (SES)

$$0 \to G_j \to P_j \to \Sigma^{2^{j+2}-1} \mathbb{Z}_2 \to 0$$

with $u_{j+2} \in \mathrm{im}(\mathrm{Sq}^2)$. Then $H_*(P_j; Q_1) = 0$, so $M_k \otimes P_j$ is a free A_1-module by Wall's Theorem ([13]), using also a Künneth Theorem for Q_i-homology. The short exact sequence of A_1-modules

$$0 \to M_k \otimes G_j \to M_k \otimes P_j \to \Sigma^{2^{j+2}-1} M_k \to 0 \tag{3.4}$$

has a long exact sequence Ext sequence which implies that

$$\mathrm{Ext}_{A_1}^{s,t}(M_k \otimes G_j, \mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s+1,t+1}(\Sigma^{2^{j+2}} M_k, \mathbb{Z}_2)$$

is bijective for $s \geq 1$ and surjective for $s = 0$. We deduce that, for the reduced submodule, $\mathrm{Ext}_{A_1}(M_k z_j, \mathbb{Z}_2)$ is formed from $\mathrm{Ext}_{A_1}(\Sigma^{2^{j+2}} M_k, \mathbb{Z}_2)$ by shifting filtrations down by 1, or, equivalently, by killing classes of filtration 0. Elements in the kernel of (3.4) when $s = 0$ correspond to free summands, which do not appear in the reduced submodule. Iterating, we have

Proposition 3.5. *For distinct $j_i \geq k-1$, $\operatorname{Ext}_{A_1}(M_k z_{j_1}\cdots z_{j_r},\mathbb{Z}_2)$ is formed from $\operatorname{Ext}_{A_1}(\Sigma^{2^{j_1+2}+\cdots+2^{j_r+2}}M_k,\mathbb{Z}_2)$ by reducing filtrations by r.*

The E_1-splitting of H^*K_2 in [8, Proposition 2.11] also involved products of modules with a class called u_2^2 there, but would be u_0^2 in our notation. Again, since $\operatorname{Sq}^2(u_0^2)\neq 0$, we must expand to an A_1-submodule of $H^*(K_2)$, namely

$$U=\langle u_0,u_1,u_0^2,u_2,u_1^2\rangle. \tag{3.6}$$

The A_1-structure of this is $\Sigma^2\langle 1,\operatorname{Sq}^1,\operatorname{Sq}^2,\operatorname{Sq}^2\operatorname{Sq}^1,\operatorname{Sq}^3\operatorname{Sq}^1\rangle$, sometimes called the Joker ([3]). Note that $H_*(U;Q_1)=\langle u_0^2\rangle$.

Proposition 3.7. *If M is a Q_0-free A_1-module and U is as above, then for $s>0$*

$$\operatorname{Ext}_{A_1}^{s,t}(U\otimes M,\mathbb{Z}_2)\approx \operatorname{Ext}_{A_1}^{s+2,t+2}(M,\mathbb{Z}_2).$$

Proof. There is a SES of A_1-modules

$$0\to G\to F\to U\to 0,$$

where F is a free A_1-module on a generator of degree 2, and $G=\langle \iota_5,\operatorname{Sq}^2\iota_5,\operatorname{Sq}^3\iota_5\rangle$. After tensoring with M, the exact Ext sequence yields an isomorphism for $s>0$

$$\operatorname{Ext}_{A_1}^{s,t}(G\otimes M,\mathbb{Z}_2)\to \operatorname{Ext}_{A_1}^{s+1,t}(U\otimes M,\mathbb{Z}_2).$$

Let $P=\Sigma^5A_1/(\operatorname{Sq}^1)$. There is a SES of A_1-modules

$$0\to \Sigma^{10}\mathbb{Z}_2\to P\to G\to 0.$$

Then $P\otimes M$ is free by Wall's theorem, since $H_*(P;Q_1)=0$ and $H_*(M;Q_0)=0$. So tensoring this sequence with M yields isomorphisms for $s>0$

$$\operatorname{Ext}_{A_1}^{s,t}(\Sigma^{10}M,\mathbb{Z}_2)\to \operatorname{Ext}_{A_1}^{s+1,t}(G\otimes M,\mathbb{Z}_2).$$

Combining the two yields

$$\operatorname{Ext}_{A_1}^{s,t}(\Sigma^{10}M,\mathbb{Z}_2)\approx \operatorname{Ext}_{A_1}^{s+2,t}(U\otimes M,\mathbb{Z}_2).$$

The Q_0-free module $U\otimes M$ has v_1^4-periodicity in Ext

$$\operatorname{Ext}_{A_1}^{s,t}(U\otimes M,\mathbb{Z}_2)\approx \operatorname{Ext}_{A_1}^{s+4,t+12}(U\otimes M,\mathbb{Z}_2)$$

for $s>0$ by [1, Theorem 5.1]. This is isomorphic to $\operatorname{Ext}_{A_1}^{s+2,t+12}(\Sigma^{10}M,\mathbb{Z}_2)\approx \operatorname{Ext}_{A_1}^{s+2,t+2}(M,\mathbb{Z}_2)$.

■

We let UM_k and $UM_kz_{j_1}\cdots z_{j_r}$ denote reduced modules after tensoring with U. By Proposition 3.7, their Ext charts are obtained from those of M_k or $M_kz_{j_1}\cdots z_{j_r}$ by decreasing filtrations by 2.

The summand S in [8, Proposition 2.11] is the reduced summand of tensor products of the summands of the type that we have been considering here with an E_1-module N with Q_1-homology class x_9. We have an analogous construction in the A_1 context.

Using the classes w_j in the proof of Theorem 2.1, let N be the A_1-module

$$N = \langle w_2, w_0w_2, w_1w_2, w_3, w_2^2\rangle.$$

This satisfies $\mathrm{Sq}^2\,\mathrm{Sq}^3(w_2) = \mathrm{Sq}^1(w_3) = w_2^2$ with $|w_2| = 5$. It has the property that if M is a Q_0-free A_1-module, then

$$\mathrm{Ext}_{A_1}(N\otimes M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}(\Sigma^9 M, \mathbb{Z}_2) \tag{3.8}$$

in positive filtration as is easily seen from the Ext sequence obtained from the SES

$$0 \to N'\otimes M \to N\otimes M \to \Sigma^9 M \to 0,$$

where N' is the A_1-submodule of N generated by w_2, since $N/N' = \Sigma^9\mathbb{Z}_2$ and $N'\otimes M$ is a free A_1-module by Wall's theorem. For any of our modules $U^\varepsilon M_kz_J$, we let $NU^\varepsilon M_kz_J$ denote a reduced submodule of $N\otimes U^\varepsilon M_kz_J$. It is isomorphic to $\Sigma^9U^\varepsilon M_kz_J$.

The analogue of [8, Proposition 2.11] is given in Theorem 3.9. We let $y_1^2 = u_0^4$; it is annihilated by Sq^1 and Sq^2.

Theorem 3.9. *There is an A_1-module splitting*

$$H^*K_2 = P[y_1^2]\otimes(\mathbb{Z}_2\oplus U\oplus N\oplus NU)\otimes(\mathbb{Z}_2\oplus\bigoplus_{k\ge4} M_k\Lambda_{k-1})\oplus F,$$

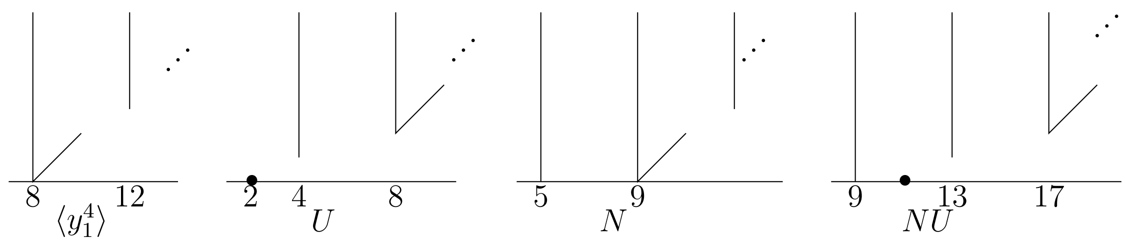
where F is free and $\Lambda_{k-1} = E[z_j : j\ge k-1]$ is an exterior algebra. The interpretation of $M_kz_{j_1}\cdots z_{j_r}$ is as in Definition 3.3, and $U\otimes M_k\Lambda_{k-1}$, $N\otimes M_k\Lambda_{k-1}$, and $NU\otimes M_k\Lambda_{k-1}$ mean the reduced summand. For reduced cohmology, one can remove the $\mathbb{Z}_2$ summand from the splitting.

Theorem 3.9 is obtained from [8, Proposition 2.11] by modifying the E_1 summands (where necessary) to make them A_1 modules that still retain the same Q_1 and Q_0 homologies.

Proof. The correspondence with [8, Proposition 2.11] is $R \leftrightarrow \bigoplus M_k\Lambda_{k-1}$, $S = NR$, $\langle u_2^2\rangle \leftrightarrow U$, and $P[u_2^2] \leftrightarrow P[y_1^2]\otimes(\mathbb{Z}_2\oplus U)$. The Q_0- and Q_1-homology classes correspond and fill out the Q_i-homology of H^*K_2. The quotient of H^*K_2 by this large submodule is A_1-free by Wall's theorem. ■

The E_2 page is obtained by applying $\mathrm{Ext}_{A_1}(-,\mathbb{Z}_2)$ to the summands of Theorem 3.9. Earlier in this section, we have done that for the summands involving $M_k\Lambda_{k-1}$. The others are small modules whose Ext is easily seen to be as in Figure 3.10.

Figure 3.10. $\mathrm{Ext}_{A_1}(-,\mathbb{Z}_2)$**.**



Here NU means the reduced summand of the A_1-module $N\otimes U$.

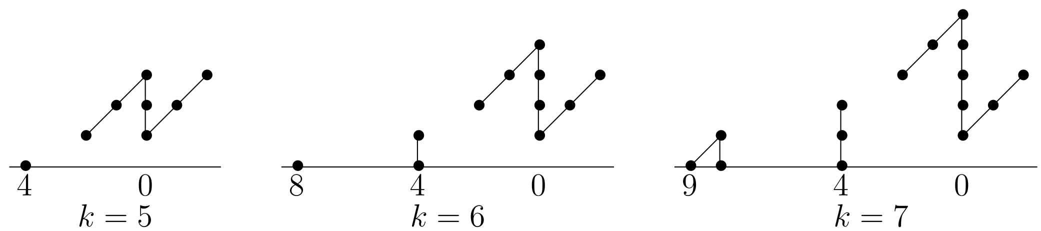
4. ko-cohomology and duality

Our main focus is on $ko_*(K_2)$, in part because of its relationship with Spin cobordism. In this short section, we explain briefly how we compute $ko^*(K_2)$ and the duality between it and $ko_*(K_2)$.

The Adams spectral sequence for $ko^{-*}(K_2)$ is obtained by applying $\mathrm{Ext}_{A_1}(\mathbb{Z}_2,-)$ to the same A_1-modules used for $ko_*(K_2)$, with corresponding differentials. As we did for ku in [8], we display the ko-cohomology groups increasing from right to left.

In Figure 4.1, we show the beginning of the charts for $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M_k)$ for $k = 5, 6, 7$. This should be enough to suggest the entire pattern. These charts are the analogue of those in Figure 3.1. They can be easily obtained from Figure 2.3.

Figure 4.1. $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, \Sigma^{-2^k} M_k)$



The analogue of Propositions 3.5 and 3.7 is as follows. It is proved using the exact sequences derived in Section 3.

Proposition 4.2. *(a). For distinct* $j_i \geq k$*,* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M_k z_{j_1} \cdots z_{j_r})$ *is formed from* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, \Sigma^{2^{j_1+2}+\cdots+2^{j_r+2}} M_k)$ *by increasing filtrations by* r *and extending to the left by* v_1^4*-periodicity.*

(b). If M *is a* Q_0*-free* A_1*-module, then* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, U \otimes M)$ *is formed from* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M)$ *by increasing fitrations by 2 and extending to the left using* v_1^4*-periodicity.*

Analogously to [8, Theorem 1.16], we have the following remarkable duality result, where the group on the right hand side is the Pontryagin dual.

Theorem 4.3. *There is an isomorphism of* ko_**-modules* $ko_*(K_2) \approx (ko^{*+6} K_2)^\vee$.

This is deduced from [11, Corollary 9.3] similarly to the ku proof in [10]. The subtlety of the result is suggested by the observation that there is nothing like it for the E_2 pages. We anticipate illustrating it in subsequent work in which differentials and extensions are determined.

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